CONTINUATION HOMOMORPHISM IN RABINOWITZ FLOER HOMOLOGY FOR SYMPLECTIC DEFORMATIONS

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Abstract. Will J. Merry computed Rabinowitz Floer homology above Mañé's critical value in terms of loop space homology in [9] by establishing an *Abbondandolo-Schwarz short exact sequence*. The purpose of this article is to provide an alternative proof of Merry's result. We construct a continuation homomorphism for symplectic deformations which enables us to reduce the computation to the untwisted case. Our construction takes advantage of a special version of the isoperimetric inequality which above Mañé's critical value holds true.

1. Introduction

Rabinowitz Floer homology as introduced in [3] is the semi-infinite dimensional Morse homology associated to Rabinowitz action functional. Critical points of Rabinowitz action functional are Reeb orbits on a fixed energy hypersurface of arbitrary period. Rabinowitz Floer homology vanishes if the energy hypersurface is displaceable, however, we have the following non-vanishing result.

Theorem 1.1 (Abbondandolo-Schwarz [1], Cieliebak-Frauenfelder-Oancea [5]). *Assume N is a closed manifold. Denote by ST^*N the unit cotangent bundle of N in the cotangent bundle T^*N which is endowed with its canonical symplectic structure. Then in degree $*\neq 0,1$*

$$\mathbf{RFH}_*(ST^*N, T^*N) = \begin{cases} \mathrm{H}_*(\mathscr{L}_N), & \textit{if} \;\; * > 1, \\ \mathrm{H}^{-*+1}(\mathscr{L}_N), & \textit{if} \;\; * < 0. \end{cases}$$

In degree 0 we have

$$\mathrm{RFH}^c_0(ST^*N, T^*N) = \begin{cases} \mathrm{H}_0(\mathscr{L}^c_N) \oplus \mathrm{H}^1(\mathscr{L}^c_N), & \textit{if} \;\; c \neq 0, \\ \mathrm{H}_0(\mathscr{L}^0_N) \oplus \mathrm{H}^1(\mathscr{L}^0_N), & \textit{if} \;\; c = 0 \textit{ and } e(T^*N) = 0, \\ \mathrm{H}^1(\mathscr{L}^0_N), & \textit{if} \;\; c = 0 \textit{ and } e(T^*N) \neq 0. \end{cases}$$

In degree 1 we have

$$\mathrm{RFH}^c_1(ST^*N, T^*N) = \begin{cases} \mathrm{H}_1(\mathscr{L}^c_N) \oplus \mathrm{H}^0(\mathscr{L}^c_N), & \textit{if} \;\; c \neq 0, \\ \mathrm{H}_1(\mathscr{L}^0_N) \oplus \mathrm{H}^0(\mathscr{L}^0_N), & \textit{if} \;\; c = 0 \textit{ and } e(T^*N) = 0, \\ \mathrm{H}_1(\mathscr{L}^0_N), & \textit{if} \;\; c = 0 \textit{ and } e(T^*N) \neq 0. \end{cases}$$

Here, $\mathscr{L}_N$ is the free loop space of N and $\mathscr{L}^c_N$ is the connected component of $\mathscr{L}_N$ of homotopy type c and $\mathrm{RFH}^c(ST^*N, T^*N)$ is the Rabinowitz Floer homology for the Rabinowitz action functional restricted to $\mathscr{L}^c_{T^*N}$. Moreover, all homology groups are taken with $\mathbb{Z}_2$-coefficients.

An interesting result of Will J. Merry tells us that this theorem continuous to hold in the presence of a weakly exact magnetic field for high enough energy levels. On the cotangent

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bundle $\tau : T^*N \to N$ of a closed Riemannian manifold (N, g), we consider an autonomous Hamiltonian system defined by a convex Hamiltonian

$$H_U(q,p) = \frac{1}{2}|p|^2 + U(q)$$

and a twisted symplectic form

$$\omega_\sigma = \omega_0 + \tau^*\sigma.$$

Here $\omega_0 = dp \wedge dq$ is the canonical symplectic form in canonical coordinates (q, p) on T^*N, $|p|$ denotes the dual norm of a Riemannian metric g on N, $U : N \to \mathbb{R}$ is a smooth potential, and σ is a closed 2-form on N. This Hamiltonian system describes the motion of a particle on N subject to the conservative force $-\nabla U$ and the magnetic field σ. We call the symplectic manifold (T^*N, ω_σ) a *twisted cotangent bundle*.

In order to state Will J. Merry's results we need the term of *Mañé critical value*. Let $(\widetilde{M}, \widetilde{g})$ be the universal cover of (M, g). Let $\sigma \in \Omega^2(M)$ denote a closed *weakly exact* 2-form, which means that the pullback $\widetilde{\sigma} \in \Omega^2(\widetilde{M})$ is exact.

Definition 1.2. Let $\sigma \in \Omega^2(N)$ be a closed weakly exact 2-form. Then the *Mañé critical value* is defined as

$$c = c(g, \sigma, U) := \inf_{\theta \in \mathscr{L}_\sigma} \sup_{q \in \widetilde{N}} \widetilde{H}_U(q, \theta_q),$$

where $\mathscr{L}_\sigma = \{\theta \in \Omega^1(\widetilde{M}) \mid d\theta = \widetilde{\sigma}\}$ and $\widetilde{H}_U$ is the lift of H_U to the universal cover.

In this article, we restrict our attention to the case of $c < \infty$ i.e. $\widetilde{\sigma} \in \Omega^2(\widetilde{N})$ admits a bounded primitive. For given $k \in \mathbb{R}$, we let $\Sigma_k := H_U^{-1}(k) \subset T^*N$. Then the dynamics of the hypersurface Σ_k changes dramatically when k is passing through c. If $k > c$ then Σ_k is *virtual restricted contact*, and Rabinowitz Floer homology is well-defined. All these things are investigated in [6]. The following theorem was conjectured in [6] and proved in [9] by using *the Abbondandolo-Schwarz short exact sequence.*

Theorem 1.3 (Merry [9])**.** *Under the above assumptions if $k > c(g, \sigma, U)$, then in degree $* \neq 0, 1$*

$$\mathbf{RFH}_*(\Sigma_k, T^*N, \omega_\sigma) = \begin{cases} \mathrm{H}_*(\mathscr{L}_N), & \text{if } * > 1, \\ \mathrm{H}^{-*+1}(\mathscr{L}_N), & \text{if } * < 0. \end{cases}$$

In degree $0, 1$ we have the same result as in Theorem 1.1.

The aim of this article is to give an alternative proof of the above theorem by constructing an explicit isomorphism between $\mathbf{RFH}(\Sigma_k, T^*N, \omega_0)$ and $\mathbf{RFH}(\Sigma_k, T^*N, \omega_\sigma)$ and then use the untwisted version, namely Theorem 1.1. The explicit isomorphism is given by the continuation homomorphism for the symplectic deformation $r \mapsto \omega_{r\sigma}$ with $r \in [0, 1]$.

Theorem 1.4. *Under the above assumptions, if $k > c(g, \sigma, U)$ and $\omega_0, \omega_\sigma \in \Omega^{\mathfrak{M}}_{\mathrm{reg}}(\Sigma_k)$ then there is a continuation map*

$$\Psi^{\omega_\sigma}_{\omega_0 *} : \mathbf{RFC}_*(\Sigma_k, \omega_0) \to \mathbf{RFC}_*(\Sigma_k, \omega_\sigma)$$

which induces an isomorphism

$$\widetilde{\Psi^{\omega_\sigma}_{\omega_0}}_* : \mathbf{RFH}_*(\Sigma_k, \omega_0) \to \mathbf{RFH}_*(\Sigma_k, \omega_\sigma).$$

One of our motivation for considering an alternative proof of Merry's result is that the continuation homomorphism can be used to compare spectral invariants between two different magnetic fields, we refer to [2] for a discussion of spectral invariants in Rabinowitz Floer homology. We plan to discuss this in more detail in a further paper.

The question of invariance under symplectic perturbation is also an important issue in symplectic homology, we refer to the paper by Ritter [13]. In view of the long exact sequence between symplectic homology and Rabinowitz Floer homology established in [5] we expect interesting interactions of this paper with the approach followed by Ritter.

2. Continuation homomorphism in Morse and Floer homology

2.1. **Morse homology.** Let (M, g) be a closed Riemannian manifold and $f : M \to \mathbb{R}$ a Morse function. We recall that the Morse chain complex $\mathrm{CM}_*(f)$ is the graded $\mathbb{Z}_2$-vector space generated by the set $\mathrm{Crit}(f)$ of critical point of f. The grading is given by the Morse index $\mu = \mu_{\mathrm{Morse}}$ of f. The boundary operator

$$\partial : \mathrm{CM}_*(f) \to \mathrm{CM}_{*-1}(f)$$

is defined on generators by counting gradient flow lines. Indeed assume that a Riemannian metric g on M satisfies the following transversality condition. Stable and unstable manifolds with respect to the negative gradient flow of $\nabla f = \nabla^g f$ intersect transversaly, that is, $W^s(x) \pitchfork W^u(y)$ for all $x, y \in \mathrm{Crit}(f)$. Then the moduli space

$$\widehat{\mathcal{M}}(x_-, x_+) := \{x : \mathbb{R} \to M \mid \partial_s x(s) + \nabla f(x(s)) = 0,\ \lim_{s\to\pm\infty} x(s) = x_\pm\}$$

is a smooth manifold of dimension $\dim \widehat{\mathcal{M}}(x_-, x_+) = \mu(x_-) - \mu(x_+)$. Moreover, $\mathbb{R}$ acts by shifting the s-coordinate. If $x_- \neq x_+$, the action is free and we denote the quotient by

$$\mathcal{M}(x_-, x_+) := \widehat{\mathcal{M}}(x_-, x_+)/\mathbb{R}.$$

Moreover, if $\mu(x_-) - \mu(x_+) = 1$ then $\mathcal{M}(x_-, x_+)$ is a finite set. Then we can define the differential $\partial = \partial(f, g)$ as a linear map which is given on generators by

$$\partial x_- := \sum_{\substack{x_+ \in \mathrm{Crit}(f) \\ \mu(x_-)-\mu(x_+)=1}} \#_2 \mathcal{M}(x_-, x_+) x_+,$$

where, $\#_2$ denotes the count of a set modulo 2. It is a deep theorem in Morse homology that the identity

$$\partial \circ \partial = 0$$

holds, see [16] for details. Then

$$\mathrm{HM}_*(f, g) := \mathrm{H}_*(\mathrm{CM}_\bullet(f), \partial(f, g))$$

is the Morse homology for the pair (f, g). Moreover, $\mathrm{HM}_*(f, g)$ equals the singular homology $\mathrm{H}_*(M)$ of M. In particular, $\mathrm{HM}(f, g)$ is independent of the choice of Morse-Smale pair (f, g).

The independence of $\mathrm{HM}(f, g)$ of Morse-Smale pair (f, g) can be shown directly using the continuation homomorphism which is constructed in the following way. For two Morse-Smale pairs $(f_\pm, g_\pm)$ we choose $T > 0$ and a smooth family $\{f_s, g_s\}_{s\in\mathbb{R}}$ of functions $f_s : M \to \mathbb{R}$ and Riemannian metrics g_s such that

$$f_s = \begin{cases} f_- & \text{for } s \leq -T \\ f_+ & \text{for } s \geq T \end{cases} \qquad g_s = \begin{cases} g_- & \text{for } s \leq -T \\ g_+ & \text{for } s \geq T. \end{cases}$$

For critical points $x_\pm \in \mathrm{Crit}(f_\pm)$, we consider the moduli space

$$\mathcal{N}(x_-,x_+)=\mathcal{N}(x_-,x_+;f_s,g_s):=\{x:\mathbb{R}\to M\mid \partial_s x(s)+\nabla^{g_s}f_s(x(s))=0,\ \lim_{s\to\pm\infty}x(s)=x_\pm\}.$$

A homotopy (f_s,g_s) is called regular if the moduli space $\mathcal{N}(x_-,x_+)$ is a smooth manifold of dimension $\dim\mathcal{N}(x_-,x_+)=\mu(x_-)-\mu(x_+)$. A generic homotopy is regular. Moreover, in the special case $f_s=f_-=f_+$ and $g_s=g_-=g_+$ we have the identity

$$\mathcal{N}(x_-,x_+)=\widehat{\mathcal{M}}(x_-,x_+). \tag{2.1}$$

If $\mu(x_-)-\mu(x_+)=0$ the space $\mathcal{N}(x_-,x_+)$ is compact. In order to verify that we need to prove a uniform energy bound of $x\in\mathcal{N}(x_-,x_+)$ as follows

$$\begin{aligned} E(x)=E_{g_s}(x)&=\int_{-\infty}^{\infty}\|\partial_s x(s)\|_{g_s}^2 ds\\ &=-\int_{-\infty}^{\infty}\langle\nabla_{g_s}f_s(x(s)),\partial_s x(s)\rangle_{g_s}ds\\ &=-\int_{-\infty}^{\infty}df_s(x(s))\partial_s x ds\\ &=-\int_{-\infty}^{\infty}\frac{d}{ds}f_s(x(s))ds+\int_{-\infty}^{\infty}\dot f_s(x(s))ds\\ &\le\|f_-\|_\infty+\|f_+\|_\infty+2T\|\dot f_s\|_\infty. \end{aligned}\tag{2.2}$$

Then we define a linear map

$$\begin{aligned} Z=Z(f_s,g_s):\mathrm{CM}_*(f_-)&\to\mathrm{CM}_*(f_+)\\ x_-&\mapsto\sum_{\substack{x_+\in\mathrm{Crit}(f_+)\\ \mu(x_-)=\mu(x_+)}}\#_2\mathcal{N}(x_-,x_+)x_+. \end{aligned}$$

We denote $\partial_\pm:=\partial(f_\pm,g_\pm)$. In the same manner as $\partial\circ\partial=0$, one proves in Morse homology

$$Z\circ\partial_-=\partial_+\circ Z,$$

see [16]. In particular, on homology level we obtain the map

$$\widetilde{Z}:\mathrm{HM}_*(f_-,g_-)\to\mathrm{HM}_*(f_+,g_+)$$

which is called the continuation homomorphism. By a homotopy-of-homotopies argument, it is proved that $\widetilde{Z}$ is independent of the chosen homotopy (f_s,g_s), see [16]. Moreover, the continuation homomorphism is functorial in the following sense. If we fix three Morse-Smale pairs (f_a,g_a), (f_b,g_b), and (f_c,g_c) we denote the corresponding continuation homomorphisms by $\widetilde{Z}_a^b:\mathrm{HM}_*(f_a,g_a)\to\mathrm{HM}_*(f_b,g_b)$ and similarly $\widetilde{Z}_a^c$ and $\widetilde{Z}_b^c$. Then we have the following identity

$$\widetilde{Z}_a^c=\widetilde{Z}_b^c\circ\widetilde{Z}_a^b.$$

Now consider the case $f_s=f_a$ and $g_s=g_a$. By (2.1), we get $\#_2\mathcal{N}(x_-,x_+)=1$ if $x_-=x_+$ and $\#_2\mathcal{N}(x_-,x_+)=0$ for the other cases. Hence we obtain the following identity

$$\widetilde{Z}_a^a=\mathrm{id}_{\mathrm{HM}_*(f_a,g_a)}.$$

In particular, we conclude that $\widetilde{Z}_a^b$ is an isomorphism with inverse $\widetilde{Z}_b^a$.

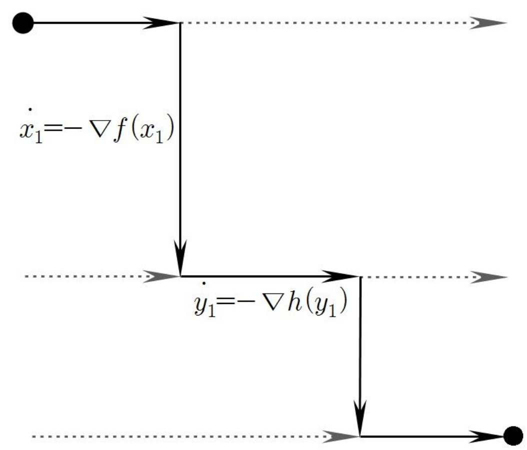

FIGURE 1. A flow line with cascades

2.2. **Morse-Bott homology.** Let M be a compact manifold and (f, h, g, g^0) be a *Morse-Bott quadruple.* The Morse-Bott quadruple consist of a Morse-Bott function f on M, a Morse function h on $\mathrm{Crit}(f)$, a Riemannian metric g on M and a Riemannian metric g^0 on $\mathrm{Crit}(f)$. We assume that (h, g^0) satisfies the Morse-Smale condition, i.e. stable and unstable manifolds intersect transversally. For a critical point c on h, let $\mathrm{ind}_f(c)$ be the number of negative eigenvalues of $\mathrm{Hess}(f)(c)$ and $\mathrm{ind}_h(c)$ be the number of negative eigenvalues of $\mathrm{Hess}(h)(c)$. We define

$$\mathrm{ind}\,(c) := \mathrm{ind}_{f,h}(c) := \mathrm{ind}_f(c) + \mathrm{ind}_h(c).$$

Definition 2.1. For $c_1, c_2 \in \mathrm{Crit}(h)$, and $m \in \mathbb{N}$ a *flow line from c_1 to c_2 with m cascades*

$$(x, T) = ((x_k)_{1\leq k\leq m}, (t_k)_{1\leq k\leq m-1})$$

consist of $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_\geq := \{r \in \mathbb{R} : r \geq 0\}$ which satisfy the following conditions:

(1) $x_k \in C^\infty(\mathbb{R}, M)$ are nonconstant solutions of
$$\dot{x}_k = -\nabla f(x_k).$$
(2) There exists $p \in W^u_h(c_1)$ and $q \in W^s_h(c_2)$ such that $\lim_{s\to-\infty} x_1(s) = p$ and $\lim_{s\to\infty} x_m(s) = q$.
(3) For $1 \leq k \leq m-1$ there are Morse flow lines $y_k \in C^\infty(\mathbb{R}, \mathrm{Crit}(f))$ of h, i.e. solutions of
$$\dot{y}_k = -\nabla h(y_k),$$
such that
$$\lim_{s\to\infty} x_k(s) = y_k(0), \qquad \lim_{s\to-\infty} x_{k+1}(s) = y_k(t_k).$$

We denote the space of flow lines with m cascades from c_1 to $c_2 \in \mathrm{Crit}(h)$ by

$$\widetilde{\mathcal{M}}_m(c_1, c_2).$$

The group $\mathbb{R}^m$ acts on $\widetilde{\mathcal{M}}_m(c_1, c_2)$ by time shift on each cascade, i.e.

$$x_k(s) \mapsto x_k(s + s_k).$$

We denote the quotient by

$$\mathcal{M}_m(c_1, c_2).$$

We define the *set of flow lines with cascades from c_1 to c_2* by

$$\mathcal{M}(c_1, c_2) := \bigcup_{m\in\mathbb{N}_0} \mathcal{M}_m(c_1, c_2).$$

For a pair (f, h) consisting of a Morse-Bott function f on M and a Morse function h on $\mathrm{Crit}(f)$, we define the chain complex $\mathrm{CM}_*(f, h)$ as the $\mathbb{Z}_2$-vector space generated by the critical points of h graded by the index. More precisely, $\mathrm{CM}_k(f, h)$ are formal sums of the form

$$\xi = \sum_{\substack{c\in\mathrm{Crit}(h)\\ \mathrm{ind}\,(c)=k}} \xi_c c$$

with $\xi_c \in \mathbb{Z}_2$. For generic choice of the Riemannian metric g on M, the moduli spaces of flow lines with cascades $\mathcal{M}(c_1, c_2)$ is a smooth manifold of dimension

$$\dim \mathcal{M}(c_1, c_2) = \mathrm{ind}\,(c_1) - \mathrm{ind}\,(c_2) - 1.$$

If $\dim \mathcal{M}(c_1, c_2) = 0$, then $\mathcal{M}(c_1, c_2)$ is finite. We define the boundary operator

$$\partial_k : \mathrm{CM}_k(f, h) \to \mathrm{CM}_{k-1}(f, h)$$

as the linear extension of

$$\partial_k c = \sum_{\mathrm{ind}\,(c')=k-1} \#_2\mathcal{M}(c, c')c'$$

for $c \in \mathrm{Crit}(h)$ with $\mathrm{ind}\,(c) = k$. The usual gluing and compactness arguments imply that

$$\partial \circ \partial = 0.$$

This defines homology groups

$$\mathrm{HM}_*(f, h, g, g^0) := \mathrm{H}_*(\mathrm{CM}_\bullet(f, h), \partial(f, h, g, g^0)).$$

In the Morse-Bott situation, we can also show that the Morse-Bott homology is independent of the choice of the Morse-Bott quadruple. First take two regular quadruples (f_-, h_-, g_-, g^0_-) and (f_+, h_+, g_+, g^0_+). Choose a smooth family of quadruples $\{(f_s, h_s, g_s, g^0_s)\}_{s\in\mathbb{R}}$ such that

$$f_s = \begin{cases} f_- \text{ for } s \le -T \\ f_+ \text{ for } s \ge T \end{cases} \qquad g_s = \begin{cases} g_- \text{ for } s \le -T \\ g_+ \text{ for } s \ge T. \end{cases}$$

$$h_s = \begin{cases} h_- \text{ for } s \le -T \\ h_+ \text{ for } s \ge T \end{cases} \qquad g^0_s = \begin{cases} g^0_- \text{ for } s \le -T \\ g^0_+ \text{ for } s \ge T. \end{cases}$$

For $c_1 \in \mathrm{Crit}(h_-)$, $c_2 \in \mathrm{Crit}(h_+)$, we consider the following flow lines from c_1 to c_2 with m cascades

$$(x, T) = ((x_k)_{1\le k\le m}, (t_k)_{1\le k\le m-1})$$

for $x_k \in C^\infty(\mathbb{R}, M)$ and $t_k \in \mathbb{R}_\ge$ which satisfy the following conditions:

(1) x_k are nonconstant solutions of

$$\dot{x}_k(s) = -\nabla_{\widetilde{g}_k} \widetilde{f}_k(x_k),$$

where

$$\widetilde{f}_k = \begin{cases} f_- & \text{for} \quad 1 \le k \le m_1 \\ f_s & \text{for} \quad k = m_1 + 1 \\ f_+ & \text{for} \quad m_1 + 2 \le k \le m \end{cases}$$

and

$$\widetilde{g}_k = \begin{cases} g_- & \text{for} \quad 1 \le k \le m_1 \\ g_s & \text{for} \quad k = m_1 + 1 \\ g_+ & \text{for} \quad m_1 + 2 \le k \le m. \end{cases}$$

(2) There exists $p_1 \in W^u_{h_-}(c_1)$ and $p_2 \in W^s_{h_+}(c_2)$ such that $\lim_{s\to-\infty} x_1(s) = p_1$ and $\lim_{s\to\infty} x_m(s) = p_2$.

(3) For $1 \le k \le m-1$, y_k are Morse flow lines of $\widetilde{h}$, i.e. solutions of

$$\dot{y}_k(s) = -\nabla_{\widetilde{g}^0_k} \widetilde{h}_k(y_k),$$

and

$$\lim_{s\to\infty} x_k(s) = y_k(0), \qquad \lim_{s\to-\infty} x_{k+1}(s) = y_k(t_k)$$

where

$$\widetilde{h}_k = \begin{cases} h_- & \text{for} \quad 1 \le k \le m_1 - 1 \\ h_s & \text{for} \quad k = m_1 \\ h_+ & \text{for} \quad m_1 + 1 \le k \le m - 1 \end{cases}$$

and

$$\widetilde{g}^0_k = \begin{cases} g^0_- & \text{for} \quad 1 \le k \le m_1 - 1 \\ g^0_s & \text{for} \quad k = m_1 \\ g^0_+ & \text{for} \quad m_1 + 1 \le k \le m - 1. \end{cases}$$

For a generic choice of the data, the space of solutions of (1) to (3) is a smooth manifold whose dimension is given by the difference of the indices of c_1 and c_2. If $\mathrm{ind}\,(c_1) = \mathrm{ind}\,(c_2)$ then this manifold is compact. In order to verify this we need to prove a uniform energy bound of time-dependent cascades as in the Morse case. Since a cascade consists of several negative gradient flow lines $(x_k)_{1\le k\le m}, (y_k)_{1\le k\le m-1}$, it suffices to show that the energy of each (time-dependent) gradient flow line are uniformly bounded. This is guaranteed by the argument of (2.2) in the Morse situation.

We define a map

$$Z = Z(\widetilde{f}, \widetilde{h}, \widetilde{g}, \widetilde{g}^0) : \mathrm{CM}_*(f_-, h_-) \to \mathrm{CM}_*(f_+, h_+)$$

as the linear extension of

$$Zc_- = \sum_{\substack{c_+ \in \mathrm{Crit}(h_+) \\ \mathrm{ind}\,(c_+) = \mathrm{ind}\,(c_-)}} \#_2 \mathcal{M}(c_-, c_+) c_+$$

where $c_- \in \mathrm{Crit}(h_-)$. Standard arguments as in the Morse case show that Z induces isomorphisms on homologies

$$\widetilde{Z} : \mathrm{HM}_*(f_-, h_-, g_-, g^0_-) \to \mathrm{HM}_*(f_+, h_+, g_+, g^0_+).$$

This proves that Morse-Bott homology is independent of the choice of a Morse-Bott quadruple. We refer to Appendix A in [7], for details.

2.3. **Floer homology for Hamiltonian deformation.** Let (M,ω) be a symplectically aspherical closed $2n$-dimensional manifold which means that $\omega|_{\pi_2(M)}\equiv 0$. Let $H: S^1\times M\to\mathbb{R}$ be a time-dependent Hamiltonian on M and $H_t = H(t,\cdot)\in C^\infty(M,\mathbb{R})$. The *Hamiltonian vector field* X_{H_t} is defined by

$$dH_t = -\iota_{X_{H_t}}\omega.$$

An almost complex structure J_t on M is ω-*compatible* if $\langle\cdot,\cdot\rangle := \omega(\cdot, J_t\cdot)$ is a Riemannian metric $\forall t\in S^1$. Let $\mathscr{L}^0$ be the component of contractible loops on M. The *Hamiltonian action* is

$$\mathcal{A}_H : \mathscr{L}^0 \to \mathbb{R}$$
$$\mathcal{A}_H(x) := \int_{\mathbb{D}^2}\overline{x}^*\omega - \int_0^1 H(t,x(t))dt,$$

where $\overline{x}$ is an extension of x to the unit disk. Since we consider only contractible loops such an extension exists and because $\omega|_{\pi_2(M)}=0$ the action functional does not depend on the choice of the filling disk. A positive gradient flow line $v:\mathbb{R}\times S^1\to M$ of $\mathcal{A}_H$ satisfies the perturbed Cauchy-Riemann equation

$$\partial_s v + J(t,v)(\partial_t v - X_H(t,v)) = 0. \tag{2.3}$$

Formally a positive gradient flow line $v\in$"$C^\infty(\mathbb{R},\mathscr{L}^0)$" is a solution of the "ODE"

$$\partial_s v - \nabla\mathcal{A}_H(v(s)) = 0.$$

According to Floer, we interpret this as a solution of the PDE, $v\in C^\infty(\mathbb{R}\times S^1, M)$ satisfying (2.3).

2.3.1. *Sign and grading conventions.* The *Conley-Zehnder index* $\mu_{\mathrm{CZ}}(x;\tau)\in\mathbb{Z}$ of a nondegenerate 1-periodic orbit x of X_H with respect to a symplectic trivialization $\tau : x^*TM\to S^1\times\mathbb{R}^{2n}$ is defined as follows. The linearized Hamiltonian flow along x with τ defines a path of symplectic matrices $\Phi_t,\ t\in[0,1]$, with $\Phi_0=\mathrm{id}$ and Φ_1 not having 1 as its spectrum. Then $\mu_{\mathrm{CZ}}(x;\tau)$ is the Maslov index of the path Φ_t as defined in [14, 15]. For a critical point x of a C^2-small Morse function H the Conley-Zehnder index with respect to the constant trivialization τ is related to the Morse index by

$$\mu_{\mathrm{CZ}}(x;\tau) = n - \mu_{\mathrm{Morse}}(x).$$

We have the following identity

$$\mu_{\mathrm{CZ}}(x;\tau') = \mu_{\mathrm{CZ}}(x;\tau) - 2c_1([\tau'\#\overline{\tau}]),$$

where c_1 is first Chern class and $\overline{\tau}$ means opposite orientation of τ. If $c_1(M)=0$ we obtain integer valued Conley-Zehnder indices for all 1-periodic orbits. Without any hypothesis on $c_1(M)$ we still have well-defined Conley-Zehnder indices in $\mathbb{Z}_2$ and all the following result hold with respect to this $\mathbb{Z}_2$-grading.

2.3.2. *Floer homology.* Let $\mathcal{P}(H)$ be the set of 1-periodic orbits of X_H. Given $x_\pm\in\mathcal{P}(H)$ we denote by $\widehat{\mathcal{M}}(x_-,x_+)$ the space of solutions of (2.3) with $\lim_{s\to\pm\infty}v(s,t)=x_\pm(t)$. Its quotient by the $\mathbb{R}$-action $s_0\cdot(s,t) := (s+s_0,t)$ on the cylinder is called the *moduli space of Floer trajectories* and is denoted by

$$\mathcal{M}(x_-,x_+) := \widehat{\mathcal{M}}(x_-,x_+)/\mathbb{R}$$

Assume now that all elements of $\mathcal{P}(H)$ are nondegenerate. Suppose further that the almost complex structure $J = (J_t),\ t \in S^1$ is generic, so that $\mathcal{M}(x_-, x_+)$ is a smooth manifold of dimension

$$\dim \mathcal{M}(x_-, x_+) = \mu_{\mathrm{CZ}}(x_-) - \mu_{\mathrm{CZ}}(x_+) - 1.$$

The boundary operator $\partial_k : \mathrm{CF}_k(H) \to \mathrm{CF}_{k-1}(H)$ is defined by

$$\partial x := \sum_{\mu_{\mathrm{CZ}}(y)=k-1} \#_2 \mathcal{M}(y, x) y.$$

It decreases the action and satisfies $\partial \circ \partial = 0$. Hence we can define Floer homology

$$\mathrm{FH}_*(H) = \mathrm{H}_*(\mathrm{CF}_\bullet(H), \partial).$$

Note that $(\mathrm{CF}_*(H), \partial)$ depends on additional data, namely the Hamiltonian H, the symplectic structure ω, and the almost complex structure J_t.

2.3.3. *Continuation map.* Now we show that $\mathrm{FH}_*(H)$ only depends only on the underlying manifold M. Take two different time-dependent Hamiltonians $H_-, H_+ \in C^\infty(S^1 \times M)$, and choose $T > 0$ and a smooth family of Hamiltonians $H_s : S^1 \times M \to \mathbb{R}$ with $s \in \mathbb{R}$ such that

$$H_s = \begin{cases} H_- \text{ for } s \leq -T \\ H_+ \text{ for } s \geq T. \end{cases}$$

Now take two different almost complex structures $J_{t,-}, J_{t,+}$, and a smooth family of almost complex structures $J_{t,s}$ such that

$$J_{t,s} = \begin{cases} J_{t,-} \text{ for } s \leq -T \\ J_{t,+} \text{ for } s \geq T. \end{cases}$$

The continuation map between two different time-dependent Hamiltonian

$$\zeta_{H_-}^{H_+} : \mathrm{CF}_*(H_-) \to \mathrm{CF}_*(H_+)$$

is given by counting positive gradient flow lines $v \in$“$C^\infty(\mathbb{R}, \mathscr{L}^0)$” of

$$\mathcal{A}_{H_s}(x) = \int_{\mathbb{D}^2} \overline{x}^* \omega - \int_0^1 H_s(t, x(t)) dt$$

where, $v \in C^\infty(\mathbb{R} \times S^1, M)$ is a solution of

$$\left.\begin{array}{l} \partial_s v + J_{t,s}(v)(\partial_t v - X_{H_s}(v)) = 0 \\ \lim_{s \to \pm\infty} v(s) = v_\pm \in \mathrm{Crit}\mathcal{A}_{H_\pm}. \end{array}\right\} \tag{2.4}$$

For critical points $v_\pm \in \mathrm{Crit}\mathcal{A}_{H_\pm}$, we consider the moduli spaces

$$\mathcal{N}_{H_\pm}(v_-, v_+) = \mathcal{N}_{H_\pm}(v_-, v_+; H_s) = \{v : \mathbb{R} \times S^1 \to M \mid v \text{ satisfies } (2.4)\}.$$

If $\mu_{\mathrm{CZ}}(v_-) = \mu_{\mathrm{CZ}}(v_+)$ the space $\mathcal{N}_{H_\pm}(v_-, v_+)$ is compact. A crucial ingredient for the compactness prove is, as in the Morse case, a uniform energy bound for $v \in \mathcal{N}_{H_\pm}(v_-, v_+)$, see [15]

for details. The uniform energy bound is given by

$$\begin{aligned}
E(v) = E_{J_{t,s}}(v) &= \int_{-\infty}^{\infty} \|\partial_s v(s)\|^2_{J_{t,s}} ds \\
&= \int_{-\infty}^{\infty} \langle \nabla^{J_{t,s}} \mathcal{A}_{H_s}(v(s)), \partial_s v(s)\rangle_{J_{t,s}} ds \\
&= \int_{-\infty}^{\infty} \frac{d}{ds}\mathcal{A}_{H_s}(v(s)) ds - \int_{-\infty}^{\infty} \dot{\mathcal{A}}_{H_s}(v(s)) ds \\
&= \mathcal{A}_{H_+}(v_+) - \mathcal{A}_{H_-}(v_-) - \int_{-\infty}^{\infty}\int_0^1 \dot{H}_s(t, v(s,t)) dt ds \\
&\leq \mathcal{A}_{H_+}(v_+) - \mathcal{A}_{H_-}(v_-) + 2T \max_{\substack{s\in[-T,T] \\ (t,x)\in S^1\times M}} |\dot{H}_s(t,x)|,
\end{aligned}$$

where $\|\cdot\|_{J_{t,s}}$ is given by $\int_0^1 \omega(\cdot, J_{t,s}\cdot)dt$. Then we can define a linear map

$$\begin{aligned}
\zeta^{H_+}_{H_-} = \zeta^{H_+}_{H_-}(H_s) : \mathrm{CF}_*(H_-) &\to \mathrm{CF}_*(H_+) \\
v_- &\mapsto \sum_{\substack{v_+\in \mathrm{Crit}\mathcal{A}_{H_+} \\ \mu_{\mathrm{CZ}}(v_-)=\mu_{\mathrm{CZ}}(v_+)}} \#_2 \mathcal{N}_{H_\pm}(v_-, v_+) v_+
\end{aligned}$$

which induces a homomorphism on homology level,

$$\widetilde{\zeta}^{H_+}_{H_-} : \mathrm{FH}_*(H_-) \to \mathrm{FH}_*(H_+).$$

The resulting homomorphism is independent of the choice of the homotopy H_s and $J_{t,s}$ by a homotopy-of-homotopies argument, similar as in the Morse situation. By functoriality, we conclude that $\widetilde{\zeta}^{H_+}_{H_-}$ is an isomorphism with inverse $\widetilde{\zeta}^{H_-}_{H_+}$.

Now consider the special case, where Hamiltonian $H \equiv 0$ is the zero Hamiltonian. Then

$$\mathcal{A}_H(x) = \mathcal{A}_0(x) = \int_{\mathbb{D}^2} \overline{x}^*\omega$$

is the symplectic area functional which is Morse-Bott and

$$\mathrm{Crit}\mathcal{A}_0 = \{x \in \mathscr{L}^0 \mid x \text{ is a constant loop}\} \cong M.$$

This implies that

$$\mathrm{FH}_*(0, f) = \mathrm{HM}_*(f) \cong \mathrm{H}_*(M),$$

where $f : M \to \mathbb{R}$ is an additional Morse function on the critical manifold $\mathrm{Crit}\mathcal{A}_0 \cong M$. Note that $\mathrm{H}_*(M)$ is the singular homology of M which only depends on M. Hence we conclude that Floer homology does not depend on additional structures like ω, H, and J_t.

2.4. **Floer homology for symplectic deformation.** In the previous subsection, we have seen that Floer homology is independent of the symplectic structure. In this subsection, we ask if this fact can also be seen directly by constructing a continuation homomorphism between two symplectic forms. So far we can only construct the continuation homomorphism for symplectic deformations under additional assumptions on the symplectic structures. Different from the case of Hamiltonian deformations, it might be necessary to subdivide the symplectic deformations in a sequence of small *adiabatic* steps.

Let (M, g) be a $2n$-dimensional closed Riemannian manifold with two symplectic forms ω_0, ω_1. Suppose that (M, ω_s) is a family of symplectically aspherical closed manifolds, where $\omega_s = s\omega_1 + (1-s)\omega_0$ for $s \in [0,1]$. Then we want to construct a continuation map

$$\Psi^{\omega_1}_{\omega_0 *} : \mathrm{CF}_*(\omega_0) \to \mathrm{CF}_*(\omega_1)$$

which induces an isomorphism on homology level. In order to state and prove our result we need the term of the *cofilling function.*

Definition 2.2 (Gromov [8], Polterovich [12])**.** Let $\sigma \in \Omega^2(M)$ be a closed weakly exact 2-form, then the *cofilling function* is

$$u_\sigma(s) : [0,\infty) \to [0,\infty)$$

$$u_\sigma(s) = u_{\sigma,g,x}(s) = \inf_{\theta \in \mathscr{L}_\sigma} \sup_{z \in B_x(s)} |\theta_z|_{\widetilde{g}},$$

where $\mathscr{L}_\sigma = \{\theta \in \Omega^1(\widetilde{M}) \mid d\theta = \widetilde{\sigma}\}$ and $B_x(s)$ be the s-ball centered at $x \in \widetilde{M}$.

Remark 2.3. If we choose another Riemannian metric g' on M and a different base point $x' \in \widetilde{M}$ then we can check that $u_{\sigma,g,x} \sim u_{\sigma,g',x'}$. [1]

Lemma 2.4 (Quadratic isoperimetric inequality)**.** Let (M, g) be a closed Riemannian manifold with closed weakly exact 2-form $\sigma \in \Omega^2(M)$. If $u_\sigma(t) \lesssim t$, then the *quadratic isoperimetric inequality* holds,

$$\int_{\mathbb{D}^2} \overline{v}^* \sigma \le C \left(l(v)^2 + 1 \right)$$

where $l(v) = \int_{S^1} |\partial_t v(t)|_g dt$, $\overline{v} : \mathbb{D}^2 \to M$ is an extension of the contractible loop $v : S^1 \to M$ to the unit disk, and $C = C(M, g, \sigma)$.

Proof. Since $u_\sigma(t) \lesssim t$, we can choose a 1-form $\theta \in \mathscr{L}_\sigma$ which has linear growth on the universal cover. Let $\widetilde{\overline{v}} : \mathbb{D}^2 \to \widetilde{M}$ be the lifting of $\overline{v}$ and set $\theta_{\max}(\widetilde{v}) = \max\limits_{z \in \widetilde{v}(S^1)} |\theta_z|_{\widetilde{g}}$. Note that

$$\begin{aligned}
\theta_{\max}(\widetilde{v}) &= \max_{z \in \widetilde{v}(S^1)} |\theta_z|_{\widetilde{g}} \\
&\le \max_{z \in B_x(l(\widetilde{v}))} |\theta_z|_{\widetilde{g}} \\
&\le 2u_\sigma(l(\widetilde{v})) \\
&\le \frac{C}{2}(l(\widetilde{v}) + 1),
\end{aligned}$$

[1] $f \sim g \iff f \lesssim g$ and $g \lesssim f$, $\quad f \lesssim g \iff \exists\, C > 0$ such that $f(s) \le C(g(s)+1),\ \forall s \in [0,\infty)$.

for some $C = C(M, g, \sigma) \in \mathbb{R}^+$. The last inequality uses the fact $u_\sigma(t) \lesssim t$. Then we get

$$\begin{aligned}\int_{\mathbb{D}^2} \overline{v}^* \sigma &= \int_{\mathbb{D}^2} \widetilde{\overline{v}}^* \widetilde{\sigma} \\ &= \int_{\mathbb{D}^2} \widetilde{\overline{v}}^* d\theta \\ &= \int_{S^1} \widetilde{v}^* \theta \\ &\leq \theta_{\max}(\widetilde{v}) l(v) \\ &\leq \frac{C}{2}\left(l(v)^2 + l(v)\right) \\ &\leq C\left(l(v)^2 + 1\right).\end{aligned}$$

Let us denote the constant C as the *isoperimetric constant.* □

Definition 2.5. Let M^{2n} be a closed manifold with a time-dependent Hamiltonian $H : S^1 \times M \to \mathbb{R}$. A pair (ω_0, ω_1) is called a *continuation pair* on (M, H) if

- (M, ω_s) is a symplectically aspherical closed manifold $\forall s \in [0, 1]$, where $\omega_s = \omega_0 + s\sigma, \quad \sigma = \omega_1 - \omega_0$;
- $\mathcal{A}_{\omega_s} = \mathcal{A}_{H,\omega_s} : \mathscr{L}^0 \to \mathbb{R}$ is Morse, for generic $s \in [0, 1]$ and $s = 0, 1$;
- $u_\sigma(t) \lesssim t$.

Remark 2.6. Let us apply Lemma 2.4 to the *continuation pair* (ω_0, ω_1) on M. First set

$$\omega_s = \omega_0 + \beta(s)\sigma, \qquad \sigma = \omega_1 - \omega_0$$

where $\beta(s) \in C^\infty(\mathbb{R}, [0, 1])$ is a cut-off function satisfying $\beta(s) = 1$ for $s \geq 1$, $\beta(s) = 0$ for $s \leq 0$ and $0 \leq \dot{\beta}(s) \leq 2$. Then we get

$$\begin{aligned}\left|\int_{\mathbb{D}^2} \overline{v}^*(\omega_s - \omega_0)\right| \leq \left|\int_{\mathbb{D}^2} \overline{v}^* \beta(s)\sigma\right| &= \beta(s)\left|\int_{\mathbb{D}^2} \overline{v}^* \sigma\right| \\ &\leq C\beta(s)\left[\left(\int_{S^1} |\partial_t v(t)| dt\right)^2 + 1\right].\end{aligned}$$

Note that $C\beta(s)$ is continuous and $C\beta(s) = 0$ for $s \leq 0$. Let us denote the function $C\beta(s)$ as the *isoperimetric constant function.*

Theorem 2.7. *Let M^{2n} be a closed manifold with a time-dependent Hamiltonian $H : S^1 \times M \to \mathbb{R}$. If (ω_0, ω_1) is a continuation pair on (M, H) then there is a continuation map*

$$\Psi^{\omega_1}_{\omega_0 *} : \mathrm{CF}_*(\omega_0) \to \mathrm{CF}_*(\omega_1)$$

which induces an isomorphism

$$\widetilde{\Psi^{\omega_1}_{\omega_0 *}} : \mathrm{FH}_*(\omega_0) \to \mathrm{FH}_*(\omega_1).$$

Proof. First recall the definition of the action functional

$$\mathcal{A}_{H,\omega} : \mathscr{L}^0 \to \mathbb{R}$$

$$\mathcal{A}_{H,\omega}(x) = \int_{\mathbb{D}^2} \overline{x}^* \omega - \int_0^1 H(t, x(t)) dt,$$

where $\overline{x}:\mathbb{D}^2\to M$ is an extension of the contractible loop x to the unit disk. By the Morse condition in the definition of the continuation pair (ω_0,ω_1), we know that $\omega_0,\omega_1\in\Omega^{\mathrm{symp}}(M)$ are nondegenerate symplectic forms. This means that every fixed point $x\in\mathrm{Fix}\phi^1_{H,\omega_i}$ is nondegenerate, where $\phi^1_{H,\omega_i}:M\to M$ is the time-1-map for the flow of the non-autonomous Hamiltonian vector field $X_H^{\omega_i}$.

Let us consider

$$\omega_s=\omega_0+\beta(s)\sigma,\qquad \sigma=\omega_1-\omega_0$$

as in Remark 2.6. We choose further almost complex structure $J_{s,t}$ for ω_s. For technical reasons, we now subdivide ω_s into sufficiently small pieces. Let $\{\omega^i\}_{i=0}^N$ be a subdivision of ω_s satisfying

- $\omega^i=\omega_0+d(i)\sigma$, where $0=d(0)<d(1)<\cdots<d(N)=1$;
- $\mathcal{A}_{H,\omega^i}:\mathscr{L}^0\to\mathbb{R}$ is Morse, $\forall i=0,1,\ldots,N$;
- $C(M,g,(d(i+1)-d(i))\sigma)\le 1/8$, $\forall i=0,1,\ldots,N-1$, where C is the isoperimetric constant.

The above 2nd condition is guaranteed by the generic Morse condition for the continuation pair (ω_0,ω_1). By Remark 2.6, we can assume the 3rd condition.

Let $\omega^i_s=\omega^i+\beta(s)(\omega^{i+1}-\omega^i)$ be a homotopy between ω^i and ω^{i+1}. Now consider $v:\mathbb{R}\times S^1\to M$ satisfying the gradient flow equation

$$\partial_s v+J_{s,t}(v)(\partial_t v-X_H^{\omega^i_s}(t,v))=0, \tag{2.5}$$

and the limit condition

$$\lim_{s\to-\infty}v(s,t)=v_-(t)\in\mathrm{Crit}\mathcal{A}_{H,\omega^i}\qquad \lim_{s\to+\infty}v(s,t)=v_+(t)\in\mathrm{Crit}\mathcal{A}_{H,\omega^{i+1}}. \tag{2.6}$$

We then want to define a map

$$\Psi^{\omega^{i+1}}_{\omega^i\ k}:\mathrm{CF}_k(\omega^i)\to\mathrm{CF}_k(\omega^{i+1})$$

given by

$$\Psi^{\omega^{i+1}}_{\omega^i\ k}(v_-)=\sum_{\mu_{\mathrm{CZ}}(v_+)=k}\#_2\mathcal{M}_{v_-,v_+}(\omega^i,\omega^{i+1})v_+.$$

Here,

$$\mathcal{M}_{v_-,v_+}(\omega^i,\omega^{i+1})=\{v:\mathbb{R}\times S^1\to M\mid v\text{ satisfies (2.5), (2.6)}\}.$$

Because ω_s is symplectically aspherical $\forall s\in\mathbb{R}$, there is no bubbling. So it suffices to bound the energy $E(v)=\int_{-\infty}^{\infty}\|\partial_s v\|_s^2ds$ of $v\in C^\infty(\mathbb{R}\times S^1,M)$ in terms of $v_-,\ v_+$ where, $\|\cdot,\cdot\|_s$ is the L^2-norm defined by $\int_0^1\omega_s(\cdot,J_s\cdot)dt$. We first compute

$$\begin{aligned}E(v)&=\int_{-\infty}^{\infty}\|\partial_s v\|_s^2ds\\&=\int_{-\infty}^{\infty}\langle\partial_s v,\nabla\mathcal{A}_{H,\omega^i_s}(v)\rangle_s ds\\&=\int_{-\infty}^{\infty}\frac{d}{ds}\mathcal{A}_{H,\omega^i_s}(v)ds-\int_{-\infty}^{\infty}\dot{\mathcal{A}}_{H,\omega^i_s}(v)ds\\&=\mathcal{A}_{H,\omega^{i+1}}(v_+)-\mathcal{A}_{H,\omega^i}(v_-)-\int_{-\infty}^{\infty}\dot{\mathcal{A}}_{H,\omega^i_s}(v)ds.\end{aligned}$$

So we need to consider the following

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} \dot{\mathcal{A}}_{H,\omega_s^i}(v)ds\right| &\leq \int_{\infty}^{\infty} \dot{\beta}(s)\left|\int_{\mathbb{D}^2} \overline{v}^*(\omega^{i+1}-\omega^i)\right| ds \\
&\leq \int_{-\infty}^{\infty} \dot{\beta}(s)C\left(\int_{S^1} |\partial_t v|_s dt\right)^2 ds + C
\end{aligned}
$$

For some $C = C(M, g, (d(i+1) - d(i))\sigma)$. Here $|\cdot,\cdot|_s$ is the norm on M induced by the Riemannian metric $\omega_s(\cdot, J_s\cdot)$. From the equation (2.5), we get

$$
\partial_t v = J(s,v)\partial_s v + X_H^{\omega_s^i}(v). \tag{2.7}
$$

By putting the above equation (2.7) into the isoperimetric inequality, we obtain

$$
\begin{aligned}
\left|\int_{-\infty}^{\infty} \dot{\mathcal{A}}_{H,\omega_s^i}(v)ds\right| &\leq \int_{-\infty}^{\infty} \dot{\beta}(s)C\left(\int_{S^1} |\partial_t v|_s dt\right)^2 ds + C \\
&\leq C\int_{-\infty}^{\infty} \dot{\beta}(s)\|\partial_t v\|_s^2 ds + C \\
&= C\int_{-\infty}^{\infty} \underbrace{\dot{\beta}(s)}_{\leq 2}\langle J_s(v)\partial_s v + X_H^{\omega_s^i}(v), J_s(v)\partial_s v + X_H^{\omega_s^i}(v)\rangle_s ds + C \\
&\leq 2C\left(\int_0^1 \|\partial_s v\|_s^2 ds + \int_0^1 \underbrace{2\langle J_s\partial_s v, X_H^{\omega_s^i}(v)\rangle_s}_{\leq \|\partial_s v\|_s^2 + \|X_H^{\omega_s^i}(v)\|_s^2} ds + \int_0^1 \|X_H^{\omega_s^i}(v)\|_s^2 ds\right) + C \\
&\leq 4C\int_{-\infty}^{\infty} \|\partial_s v\|_s^2 ds + 4C\int_0^1 \|X_H^{\omega_s^i}(v)\|_s^2 ds + C \\
&\leq 4C\ E(v) + 4C\ c' + C,
\end{aligned}
$$

where $c' \in \mathbb{R}$ is choosen satisfying $\|X_H^{\omega_s^i}(v)\|_s^2 \leq c'$. This is possible by the compactness of M. Thus we get

$$
E(v) \leq \underbrace{\mathcal{A}_{H,\omega^{i+1}}(v_+) - \mathcal{A}_{H,\omega^i}(v_-) + 4Cc' + C}_{=:c''} + 4C\ E(v).
$$

Since $C = C(M, g, (d(i+1) - d(i))\sigma) \leq \frac{1}{8}$, we finally obtain

$$
\begin{aligned}
E(v) &\leq c'' + \frac{1}{2}E(v) \\
E(v) &\leq 2c''.
\end{aligned}
$$

Now we define the continuation map from ω_0 to ω_1 by juxtaposition

$$
\Psi_{\omega_0}^{\omega_1} : \mathrm{CF}(\omega_0) \to \mathrm{CF}(\omega_1)
$$

$$
\Psi_{\omega_0}^{\omega_1} = \Psi_{\omega^{N-1}}^{\omega^N} \circ \cdots \circ \Psi_{\omega^1}^{\omega^2} \circ \Psi_{\omega^0}^{\omega^1}.
$$

By a standard argument in Floer homology theory, each $\Psi_{\omega^i}^{\omega^{i+1}}$ commutes with the boundary operators of the Floer chain complex. This implies that $\Psi_{\omega_0}^{\omega_1}$ also interchanges the boundary

operators. Hence we get an induced homomorphism

$$\widetilde{\Psi_{\omega_0}^{\omega_1}} : \mathrm{FH}(M,\omega_0) \to \mathrm{FH}(M,\omega_1).$$

In a similar way we can construct

$$\widetilde{\Psi_{\omega_1}^{\omega_0}} : \mathrm{FH}(M,\omega_1) \to \mathrm{FH}(M,\omega_0),$$

by following the homotopy backwards. By a homotopy-of-homotopies argument, we conclude $\widetilde{\Psi_{\omega_1}^{\omega_0}} \circ \widetilde{\Psi_{\omega_0}^{\omega_1}} = \mathrm{id}_{\mathrm{FH}(M,\omega_0)}$ and $\widetilde{\Psi_{\omega_0}^{\omega_1}} \circ \widetilde{\Psi_{\omega_1}^{\omega_0}} = \mathrm{id}_{\mathrm{FH}(M,\omega_1)}$. Therefore $\widetilde{\Psi_{\omega_0}^{\omega_1}}$ is an isomorphism with inverse $\widetilde{\Psi_{\omega_1}^{\omega_0}}$. □

Remark 2.8. In the proof of Theorem 2.7, the *quadratic isoperimetric inequality* is essential. One can check that if $u_{\omega_1-\omega_0}(t) \gtrsim t$, the above proof does not work anymore.

3. Rabinowitz Floer homology

3.1. **RFH for the cotangent bundle endowed with its canonical symplectic form.** In this section, we consider the cotangent bundle $(T^*N, \omega_0 = d\lambda_{\mathrm{liou}})$ of a closed Riemannian manifold (N,g) where $\lambda_{\mathrm{liou}} = p \wedge dq$ is the *Liouville 1-form* for canonical coordinates $(q,p) \in T^*N$. On the exact symplectic manifold $(T^*N, \lambda_{\mathrm{liou}})$, the *Liouville vector field* X is defined by $\iota_X\omega_0 = \lambda_{\mathrm{liou}}$. $(T^*N, \lambda_{\mathrm{liou}})$ is *complete and convex* i.e. the following conditions hold:

- There exists a compact subset $K \in T^*N$ with smooth boundary such that X points out of K along ∂K;
- The vector field X is complete and has no critical point outside K.

Equivalently, $(T^*N, \lambda_{\mathrm{liou}})$ is complete and convex since there exists an embedding $\phi : \Sigma \times [1,\infty) \to T^*N$ such that $\phi^*\lambda = r\alpha_\Sigma$, where r denotes the coordinates on $[1,\infty)$ and α_Σ is a contact form, and such that $T^*N \setminus \phi(\Sigma \times (1,\infty))$ is compact.

Consider now a complete convex exact symplectic manifold $(T^*N, \lambda_{\mathrm{liou}})$ and a compact subset $DT^*N \subset T^*N$ with smooth boundary $\Sigma := ST^*N = \partial DT^*N$ such that $\lambda_{\mathrm{liou}}|_{ST^*N}$ is a positive contact form with a Reeb vector field R. We abbreviate by $\mathscr{L} := \mathscr{L}_{T^*N} = C^\infty(S^1, T^*N)$ the free loop space of T^*N. A *defining Hamiltonian* for Σ is a smooth function $H : T^*N \to \mathbb{R}$ with regular level set $\Sigma = H^{-1}(0)$ whose *Hamiltonian vector field* X_H has compact support and agrees with R along Σ. Given such a Hamiltonian, the *Rabinowitz action functional* is defined by

$$\mathcal{A}_H : \mathscr{L} \times \mathbb{R} \to \mathbb{R}$$
$$\mathcal{A}_H(x,\eta) := \int_0^1 x^*\lambda - \eta \int_0^1 H(x(t))dt.$$

Critical points of $\mathcal{A}_H$ are solutions of the equations

$$\left.\begin{array}{l} \partial_t x(t) = \eta X_H(x(t)), \quad t \in \mathbb{R}/\mathbb{Z} \\ \int_0^1 H(x(t))dt = 0. \end{array}\right\} \tag{3.1}$$

By the first equation H is constant along x, so the second equation implies $H(x(t)) \equiv 0$. Since $X_H = R$ along Σ, the equations (3.1) are equivalent to

$$\left.\begin{array}{rl} \partial_t x(t) = \eta R(x(t)), & t \in \mathbb{R}/\mathbb{Z} \\ x(t) \in \Sigma, & t \in \mathbb{R}/\mathbb{Z}. \end{array}\right\}$$

So there are three types of critical points i.e. closed Reeb orbits on Σ:

- Positively parametrized closed Reeb orbits corresponding to $\eta > 0$;

- Negatively parametrized closed Reeb orbits corresponding to $\eta < 0$;
- Constant loops on M which corresponding to $\eta = 0$.

The action of a critical point (x, η) is $\mathcal{A}_H(s, \eta) = \eta$.

A compatible almost complex structure J on part of the symplectization $(\Sigma \times \mathbb{R}_+, d(r\alpha_\Sigma))$ of a contact manifold (Σ, α_Σ) is called *cylindrical* if it satisfies:

- J maps the Liouville vector field $r\partial_r$ to the Reeb vector field R;
- J preserves the contact distribution $\ker \alpha_\Sigma$;
- J is invariant under the Liouville flow $(y, r) \mapsto (y, e^t r),\ t \in \mathbb{R}$.

For a smooth family $(J_t)_{t \in S^1}$ of cylindrical almost complex structures on $(T^*N, \lambda_{\text{liou}})$ we consider the following metric $g = g_J$ on $\mathscr{L} \times \mathbb{R}$. Given a point $(x, \eta) \in \mathscr{L} \times \mathbb{R}$ and two tangent vectors $(\hat{x}_1, \hat{\eta}_1),\ (\hat{x}_2, \hat{\eta}_2) \in T_{(x,\eta)}(\mathscr{L} \times \mathbb{R}) = \Gamma(S^1, x^*T(T^*N)) \times \mathbb{R}$ the metric is given by

$$g_{(x,\eta)}((\hat{x}_1, \hat{\eta}_1), (\hat{x}_2, \hat{\eta}_2)) = \int_0^1 \omega\left(\hat{x}_1, J_t(x(t))\hat{x}_2\right) dt + \hat{\eta}_1 \cdot \hat{\eta}_2.$$

The gradient of the Rabinowitz action functional $\mathcal{A}_H$ with respect to the metric g_J at a point $(x, \eta) \in \mathscr{L} \times \mathbb{R}$ reads

$$\nabla \mathcal{A}_H(x, \eta) = \nabla_J \mathcal{A}_H(x, \eta) = \begin{pmatrix} -J_t(x)\left(\partial_t x - \eta X_H(x)\right) \\ -\int_0^1 H(x(t))dt \end{pmatrix}.$$

Hence the positive gradient flow lines are solutions $(x, \eta) \in C^\infty(\mathbb{R} \times S^1, T^*N) \times C^\infty(\mathbb{R}, \mathbb{R})$ of the partial differential equation

$$\left.\begin{aligned} \partial_s x + J_t(x)\left(\partial_t x - \eta X_H(x)\right) = 0 \\ \partial_s \eta + \int_0^1 H(x(t))dt = 0 \end{aligned}\right\}.$$

Then for $-\infty < a < b \le \infty$ the resulting truncated Floer homology groups

$$\mathbf{RFH}^{(a,b)}(\Sigma, T^*N) := \mathrm{HM}^{(a,b)}(\mathcal{A}_H, J),$$

corresponding to action values in (a, b), are well-defined and do not depend on the choice of the cylindrical J and the defining Hamiltonian H. The *Rabinowitz Floer homology* of (Σ, T^*N) is defined as the limit

$$\mathbf{RFH}_*(\Sigma, T^*N) := \varinjlim_a \varprojlim_b \mathbf{RFH}_*^{(-a,b)}(\Sigma, T^*N), \qquad a, b \to \infty.$$

This definition is equivalent to the original one in [3] by [4, Theorem A].

Since the Rabinowitz action functional is defined on the full loop space and the first part of the differential in the Rabinowitz Floer complex counts topological cylinders, we can split the Rabinowitz Floer homology into factors labeled by free homotopy classes

$$\mathbf{RFH}(\Sigma, T^*N) = \bigoplus_{\nu \in [S^1, T^*N]} \mathrm{RFH}^\nu(\Sigma, T^*N),$$

where $\mathrm{RFH}^\nu(\Sigma, T^*N)$ is the Rabinowitz Floer homology for the Rabinowitz action functional restricted to $\mathscr{L}^\nu = \mathscr{L}^\nu_{T^*N}$.

3.1.1. *Index and grading convention.* Let $\mathcal{M}$ be the moduli space of all finite energy gradient flow lines of the action functional $\mathcal{A}_H : \mathscr{L} \times \mathbb{R} \to \mathbb{R}$. Since $\mathcal{A}_H$ is Morse-Bott, every finite energy gradient flow line $(v,\eta) \in C^\infty(\mathbb{R}\times S^1, V) \times C^\infty(\mathbb{R},\mathbb{R})$ converges exponentially at both ends to critical points $(v_\pm, \eta_\pm) \in \mathrm{Crit}(\mathcal{A}_H)$ as the flow parameter goes to $\pm\infty$. The linearization of the gradient flow equation along any path (v,η) in $\mathscr{L}\times\mathbb{R}$ which converges exponentially to the critical point of $\mathcal{A}_H$ gives rise to a Fredholm operator $D^{\mathcal{A}_H}_{(v,\eta)}$. Let $C^-, C^+ \subset \mathrm{Crit}(\mathcal{A}_H)$ be the connected component of the critical manifold of $\mathcal{A}_H$ containing $(v_-,\eta_-), (v_+,\eta_+)$ respectively. The local virtual dimension of $\mathcal{M}$ at a finite energy gradient flow line is defined to be

$$\mathrm{virdim}_{(v,\eta)}\mathcal{M} := \mathrm{ind}\, D^{\mathcal{A}_H}_{(v,\eta)} + \dim C^- + \dim C^+$$

where $\mathrm{ind}\, D^{\mathcal{A}_H}_{(v,\eta)}$ is the Fredholm index of the Fredholm operator $\mathrm{ind}\, D^{\mathcal{A}_H}_{(v,\eta)}$. For generic compatible almost complex structures, the moduli space of finite energy gradient flow lines is a manifold and the local virtual dimension of the moduli space at a gradient flow line (v,η) corresponds to the dimension of the connected component of $\mathcal{M}$ containing (v,η).

To define a $\mathbb{Z}$-grading on $\mathbf{RFH}(\Sigma, T^*N)$, we need that the local virtual dimension just depends on the asymptotics of the finite energy gradient flow line. Since $I_{c_1} = 0$ on T^*N, it can be shown that the local virtual dimension equals

$$\mathrm{virdim}_{(v,\eta)}\mathcal{M} = \mu_{\mathrm{CZ}}(v_+) - \mu_{\mathrm{CZ}}(v_-) + \frac{\dim C^- + \dim C^+}{2}.$$

In order to deal with the third term it is useful to introduce the following index for the Morse function h on $\mathrm{Crit}(\mathcal{A}_H)$. We define the *signature index* $\mathrm{ind}^\sigma_h(c)$ of a critical point c of h to be

$$\mathrm{ind}^\sigma_h(c) := -\frac{1}{2}\mathrm{sign}(\mathrm{Hess}_h(c)).$$

We define a *grading* μ on $\mathbf{RFC}(\Sigma, T^*N) = \mathrm{CM}(\mathcal{A}_H, h)$ by

$$\mu(c) := \mu_{\mathrm{CZ}}(c) + \mathrm{ind}^\sigma_h(c) + \frac{1}{2}.$$

These define a $\mathbb{Z}$-grading on the homology $\mathbf{RFH}(\Sigma, T^*N)$. We refer to [3, 10] for more details.

3.2. **Rabinowitz Floer homology for a twisted cotangent bundle.** In the previous section, we considered an exact symplectic manifold. By the exactness of symplectic form, there is no need to care about the filling disk of a given loop. In general, twisted symplectic forms are not exact. In order to define Rabinowitz Floer homology, we need the notions of a *symplectically atoroidal manifold* and a *virtual restricted contact type hypersurface.*

Definition 3.1. A symplectic manifold (M,ω) is called *symplectically atoroidal* if

$$\int_{\mathbb{T}^2} f^*\omega = 0,$$

for any smooth function $f : \mathbb{T}^2 \to T^*N$.

Remark 3.2. Since there is a surjective map $g : \mathbb{T}^2 \to S^2$, *symplectically atoroidal* implies *symplectically aspherical.*

Lemma 3.3 (Merry [9])**.** Let $\sigma \in \Omega^2(N)$ be a weakly exact 2-form and $u_\sigma \sim 1$, then $f^*\sigma$ is exact for any smooth map $f : \mathbb{T}^2 \to N$.

Proof. Consider $G := f_*(\pi_1(\mathbb{T}^2)) \leq \pi_1(N)$. Then G is amenable, since $\pi_1(\mathbb{T}^2) = \mathbb{Z}^2$, which is amenable. Now Lemma 5.3 in [11] tells us that since $\|\theta\|_\infty < \infty$, we can replace θ by a G-invariant primitive θ' of $\widetilde{\sigma}$, which descends to a primitive $\theta'' \in \Omega^1(\mathbb{T}^2)$ of $f^*\sigma$. □

Remark 3.4. Given a free homotopy class $\nu \in [S^1, T^*N]$, fix a reference loop $v_\nu = (q_\nu, p_\nu) \in \mathscr{L}^\nu_{T^*N}$. Let D be a cylinder $S^1 \times [0,1]$ with two boundary components $\partial' D$ with the boundary orientation and $\partial'' D$ with the opposite boundary orientation. By abuse of notation, let $\overline{v} : D \to T^*N$ denote any smooth map such that $\overline{v}|_{\partial' D} = v$ and $\overline{v}|_{\partial'' D} = v_\nu$. Then thanks to the previous lemma, the integral $\int_D \overline{v}^*\tau^*\sigma$ is independent of the choice of $\overline{v}$. Similarly given any $q \in \mathscr{L}^{\tau\nu}_N$, let $\overline{q} : D \to N$ denote any smooth map such that $\overline{q}|_{\partial' D} = q$ and $\overline{q}|_{\partial'' D} = q_\nu$. Then the integral $\int_D \overline{q}^*\sigma$ is independent of the choice of $\overline{q}$. Note that in particular if $q = \tau \circ v$ then

$$\int_D \overline{v}^*\tau^*\sigma = \int_D \overline{q}^*\sigma.$$

In particular, let $\sigma \in \Omega^2(N)$ be a weakly exact 2-form satisfying $u_\sigma \sim 1$, then the twisted cotangent bundle (T^*N, ω_σ) is a *symplectically atoroidal manifold*. Moreover, the Rabinowitz action functional

$$\mathcal{A}_{\omega_\sigma} : \mathscr{L} \times \mathbb{R} \to \mathbb{R}$$

$$\mathcal{A}_{\omega_\sigma}(v, \eta) := \mathcal{A}_{H,\omega_\sigma}(v, \eta) = \int_D \overline{v}^*\omega_\sigma - \eta \int_0^1 H(v(t))dt$$

is well-defined, independent of the choice of $\overline{v}$.

Definition 3.5. A closed hypersurface Σ in a symplectic manifold (M, ω) is called *virtually contact*, if there is a covering $p : \widehat{M} \to M$ and a primitive $\lambda \in \Omega^1(\widehat{\Sigma})$ of $p^*\omega$ such that

$$\sup_{x \in \widehat{\Sigma}} |\lambda_x| \leq C < \infty, \qquad \inf_{x \in \widehat{\Sigma}} \lambda(R) \geq \mu > 0, \tag{3.2}$$

where $|\cdot|$ is the lifting of a metric on Σ and R is the pullback of a vector field generating $\ker(\omega|_\Sigma)$.

Definition 3.6. A closed hypersurface Σ in a symplectic manifold (M, ω) is called *virtual restricted contact*, if there is a covering $p : \widehat{M} \to M$ and a primitive $\lambda \in \Omega^1(\widehat{M})$ of $p^*\omega$ such that λ satisfy (3.2) again on $\widehat{\Sigma}$.

Remark 3.7. A *virtual restricted contact homotopy* is a smooth homotopy $(\Sigma_t, \lambda_t) \subset (M, \omega)$ of virtual restricted contact hypersurfaces with the corresponding 1-forms on the covers such that the preceding conditions hold with constants C, μ independent of t. $\mathrm{RFH}(\Sigma, M)$ is defined for each virtual restricted contact hypersurface Σ and is invariant under virtual restricted contact homotopies. For a twisted cotangent bundle (T^*N, ω_σ) with any $k \in \mathbb{R}$ above Mañé ciritical value $c = c(g, \sigma, U)$ the hypersurface $\Sigma_k = H_U^{-1}(k) \subset T^*N$ is virtual restricted contact, see [6].

4. Continuation homomorphism in RFH for symplectic deformations

Let us begin with the *defining Hamiltonian* H of the virtual restricted contact hypersurface $\Sigma_k \subset T^*N$

$$H := H_{U,k,\xi} = \beta_\xi \circ (H_U - k)$$

where, $\beta_\xi(t)$ is a smooth cut-off function satisfying $0 \leq \dot{\beta}_\xi \leq 1$,

$$\beta_\xi(t) = \begin{cases} t & \text{if} \quad |t| \leq \xi - \epsilon \\ \xi & \text{if} \quad t \geq \xi + \epsilon \\ -\xi & \text{if} \quad -t \geq \xi + \epsilon \end{cases}, \qquad \epsilon = \min\{1/3, \xi/3\}.$$

Now we define the Rabinowitz action functional given by

$$\mathcal{A}_{\omega_\sigma} : \mathscr{L} \times \mathbb{R} \to \mathbb{R}$$

$$\mathcal{A}_{\omega_\sigma}(v, \eta) := \mathcal{A}_{H,\omega_\sigma}(v, \eta) = \int_D \overline{v}^* \omega_\sigma - \eta \int_0^1 H(v(t))dt,$$

where $\overline{v}, D$ are given in Remark 3.4.

In this section, we consider the canonical cotangent bundle (T^*N, ω_0) and the twisted cotangent bundle (T^*N, ω_σ) with the virtual restricted contact hypersurface $\Sigma_k = H^{-1}(0) = H_U^{-1}(k)$ where $k > c(g, \sigma, U)$ and $H_U(q, p) = \frac{1}{2}|p|_g^2 + U(q)$. For convenience, let us define the following sets

$$\begin{aligned}
&\mathfrak{M}(N) = \{\sigma \in \Omega^2(N) \mid \widetilde{\sigma} = d\theta,\ \|\theta\|_\infty < \infty\};\\
&\Omega^{\mathfrak{M}}(T^*N) = \{\omega_\sigma \in \Omega^2(T^*N) \mid \sigma \in \mathfrak{M}(N)\};\\
&\Omega^{\mathfrak{M}}(\Sigma_k) = \{\omega_\sigma \in \Omega^{\mathfrak{M}}(T^*N) \mid k > c(g, \sigma, U)\};\\
&\Omega^{\mathfrak{M}}_{\text{reg}}(\Sigma_k) = \{\omega_\sigma \in \Omega^{\mathfrak{M}}(\Sigma_k) \mid \mathcal{A}_{\omega_\sigma} : \mathscr{L} \times \mathbb{R} \to \mathbb{R} \text{ is Morse-Bott}\}.
\end{aligned}$$

Note that $\Omega^{\mathfrak{M}}(\Sigma_k)$ is convex and $\Omega^{\mathfrak{M}}_{\text{reg}}(\Sigma_k)$ is dense in $\Omega^{\mathfrak{M}}(\Sigma_k)$.

For a pair $(\omega_0, \omega_\sigma)$ of $\Omega^{\mathfrak{M}}_{\text{reg}}(\Sigma_k)$, we construct the continuation homomorphism

$$\widetilde{\Psi^{\omega_\sigma}_{\omega_0 *}} : \mathbf{RFH}_*(\Sigma_k, \omega_0) \to \mathbf{RFH}_*(\Sigma_k, \omega_\sigma),$$

by counting solutions of an s-dependent Rabinowitz Floer equation. Before the construction, we must check the L_∞-bound of the Lagrange multiplier η in the case of a twisted cotangent bundle with virtual restricted contact hypersurface. The proof of the following proposition proceeds as [3] for the restricted contact type case. It was already used with no explicit proof in [6]. For the readers convenience we include a proof here.

Proposition 4.1. *Let (T^*N, ω_σ) be a twisted cotangent bundle with a virtual restricted contact hypersurface $\Sigma_k = H^{-1}(0)$ where $k > c$. Then there exist constants $\epsilon > 0$ and $\overline{c} < \infty$ such that the following holds*

$$\|\nabla \mathcal{A}_{\omega_\sigma}(v, \eta)\| \leq \epsilon \Longrightarrow |\eta| \leq \overline{c}(|\mathcal{A}_{\omega_\sigma}(v, \eta)| + 1).$$

Lemma 4.2. Under the same assumptions as in Proposition 4.1 for $(v, \eta) \in \operatorname{Crit}(\mathcal{A}_{\omega_\sigma}|_{\mathscr{L}^0 \times \mathbb{R}})$ we have

$$|\mathcal{A}_{\omega_\sigma}(v, \eta)| \geq \frac{|\eta|}{c'}$$

where $c' > 0$.

PROOF. Inserting (3.1) into $\mathcal{A}_{\omega_\sigma}$ and using the assumption of virtual restricted contact type

$$\begin{aligned} |\mathcal{A}_{\omega_\sigma}(v,\eta)| &= \left|\int_D \overline{v}^*\omega_\sigma\right| = \left|\int_D \tilde{\overline{v}}^*\pi^*\omega_\sigma\right| = \left|\int_D \tilde{\overline{v}}^* d\lambda_\sigma\right| = \left|\int_{S^1} \tilde{v}^*\lambda_\sigma\right| \\ &= \left|\int_0^1 \lambda_\sigma(\partial_t\tilde{v})\right| = \left|\int_0^1 \lambda_\sigma(\eta\widetilde{X_H^{\omega_\sigma}}(\tilde{v}))\right| \\ &= \left|\eta\int_0^1 \lambda_\sigma(\widetilde{X_H^{\omega_\sigma}}(\tilde{v}))\right| \\ &\geq \frac{|\eta|}{c'} \end{aligned} \tag{4.1}$$

where $\tilde{\overline{v}},\ \widetilde{X_H^{\omega_\sigma}}$ are lifts of $\overline{v},\ X_H^{\omega_\sigma}$ respectively to the cover $\pi:\widetilde{\Sigma}\to\Sigma$. The constant $c'>0$ exists by the second inequality in (3.2). □

Proof of Proposition 4.1. The proof consists of 3 steps. First fix $\nu\in[S^1,T^*N]$.

Step 1 : *There exist $\delta>0$ and a constant $c_\delta<\infty$ with the following property. For every $(v,\eta)\in\mathscr{L}^\nu\times\mathbb{R}$ such that $v(t)\in U_\delta=H^{-1}(-\delta,\delta)$ for every $t\in\mathbb{R}/\mathbb{Z}$, the follwing estimate holds:*

$$|\eta|\leq 2c'|\mathcal{A}_{\omega_\sigma}(v,\eta)|+c_\delta\|\nabla\mathcal{A}_{\omega_\sigma}(v,\eta)\|+2c'|a^\nu|.$$

Let c' as in (4.1). Choose $\delta>0$ so small such that

$$\lambda_\sigma(x)\widetilde{X_H^{\omega_\sigma}}(x)\geq\frac{1}{2c'}+\delta,\quad x\in\pi^{-1}(U_\delta).$$

and

$$c_\delta=2c'\|\lambda_\sigma|_{\pi^{-1}(U_\delta)}\|_\infty<\infty. \tag{4.2}$$

That (4.2) holds for sufficiently small δ is guaranteed by the first inequality in (3.2). We estimate

$$\begin{aligned} |\mathcal{A}_{\omega_\sigma}(v,\eta)| &= \left|\int_0^1\lambda_\sigma(\tilde{v})(\partial_t\tilde{v})-\underbrace{\int_0^1\lambda_\sigma(\tilde{v}_\nu)(\partial_t\tilde{v}_\nu)}_{:=a^\nu}-\eta\int_0^1 H(v(t))dt\right| \\ &= \left|\eta\int_0^1\lambda_\sigma(\tilde{v})(\widetilde{X_H^{\omega_\sigma}}(\tilde{v}))+\int_0^1\lambda_\sigma(\tilde{v})(\partial_t\tilde{v}-\eta\widetilde{X_H^{\omega_\sigma}}(\tilde{v}))-a^\nu-\eta\int_0^1 H(v(t))dt\right| \\ &\geq \left|\eta\int_0^1\lambda_\sigma(\tilde{v})(\widetilde{X_H^{\omega_\sigma}}(\tilde{v}))\right|-\left|\int_0^1\lambda_\sigma(\tilde{v})(\partial_t\tilde{v}-\eta\widetilde{X_H^{\omega_\sigma}}(\tilde{v}))\right|-\left|\eta\int_0^1 H(v(t))dt\right|-|a^\nu| \\ &\geq |\eta|\left(\frac{1}{2c'}+\delta\right)-\frac{c_\delta}{2c'}\|\partial_t\tilde{v}-\eta\widetilde{X_H^{\omega_\sigma}}(\tilde{v})\|_1-|\eta|\delta-|a^\nu| \\ &\geq \frac{|\eta|}{2c'}-\frac{c_\delta}{2c'}\|\partial_t\tilde{v}-\eta\widetilde{X_H^{\omega_\sigma}}(\tilde{v})\|_2-|a^\nu| \\ &\geq \frac{|\eta|}{2c'}-\frac{c_\delta}{2c'}\|\nabla\mathcal{A}_{\omega_\sigma}(v,\eta)\|-|a^\nu|, \end{aligned}$$

where $v_\nu\in\mathscr{L}^\nu$ is a reference loop defined in Remark 3.4. This proves Step 1.

Step 2 : *For each $\delta>0$, there exists $\epsilon=\epsilon(\delta)>0$ such that if $\|\nabla\mathcal{A}_{\omega_\sigma}(v,\eta)\|\leq\epsilon$ then $v(t)\in U_\delta$ for every $t\in[0,1]$.*

First assume that $v \in \mathscr{L}$ has the property that there exist $t_0, t_1 \in \mathbb{R}/\mathbb{Z}$ such that $|H(v(t_0))| \geq \delta$ and $|H(v(t_1))| \leq \delta/2$. We claim that

$$\|\nabla \mathcal{A}_{\omega_\sigma}(v,\eta)\| \geq \frac{\delta}{2\kappa} \tag{4.3}$$

for every $\eta \in \mathbb{R}$, where

$$\kappa := \max_{x \in \overline{U}_\delta, t \in S^1} \|\nabla_t H(x)\|.$$

To see this, assume without loss of generality that $t_0 < t_1$ and $\delta/2 \leq |H(v(t))| \leq \delta$ for all $t \in [t_0, t_1]$. Then we estimate

$$\begin{aligned}
\|\nabla \mathcal{A}_{\omega_\sigma}(v,\eta)\| &\geq \sqrt{\int_0^1 \|\partial_t v - \eta X_H^{\omega_\sigma}(v)\|^2 dt} \\
&\geq \int_0^1 \|\partial_t v - \eta X_H^{\omega_\sigma}(v)\| dt \\
&\geq \int_{t_0}^{t_1} \|\partial_t v - \eta X_H^{\omega_\sigma}(v)\| dt \\
&\geq \frac{1}{\kappa} \int_{t_0}^{t_1} \|\nabla H(v)\| \cdot \|\partial_t v - \eta X_H^{\omega_\sigma}(v)\| dt \\
&\geq \frac{1}{\kappa} \int_{t_0}^{t_1} |\langle \nabla H(v), \partial_t v - \eta X_H^{\omega_\sigma}(v)\rangle| dt \\
&\geq \frac{1}{\kappa} \int_{t_0}^{t_1} |\langle \nabla H(v), \partial_t v\rangle| dt \\
&\geq \frac{1}{\kappa} \int_{t_0}^{t_1} |dH(v)\partial_t v| dt \\
&\geq \frac{1}{\kappa} \int_{t_0}^{t_1} |\partial_t H(v)| dt \\
&\geq \frac{1}{\kappa} \left| \int_{t_0}^{t_1} \partial_t H(v) dt \right| \\
&\geq \frac{1}{\kappa} |H(v(t_1)) - H(v(t_0))| \\
&\geq \frac{1}{\kappa} \left(|H(v(t_1))| - |H(v(t_0))|\right) \\
&\geq \frac{\delta}{2\kappa}.
\end{aligned}$$

Now assume that $v \in \mathscr{L}$ has the property that $v(t) \in T^*N \setminus U_{\delta/2}$ for every $t \in [0,1]$. In this case we estimate

$$\|\nabla \mathcal{A}_{\omega_\sigma}(v,\eta)\| \geq \left| \int_0^1 H(v(t)) dt \right| \geq \frac{\delta}{2} \tag{4.4}$$

for every $\eta \in \mathbb{R}$. From (4.3) and (4.4) Step 2 follows with

$$\epsilon = \frac{\delta}{2 \max\{1, \kappa\}}.$$

Step 3 : *We prove the proposition.* Choose δ as in Step 1, $\epsilon = \epsilon(\delta)$ as in Step 2 and

$$\bar{c} = \max\{2c', 2c_\delta\epsilon, 4c'|a^\nu|\}.$$

Assume that $\|\nabla\mathcal{A}_{\omega_\sigma}(v,\eta)\| \le \epsilon$ then

$$|\eta| \le 2c'|\mathcal{A}_{\omega_\sigma}(v,\eta)| + c_\delta\|\nabla\mathcal{A}_{\omega_\sigma}(v,\eta)\| + 2c'|a^\nu| \le \bar{c}(|\mathcal{A}_{\omega_\sigma}(v,\eta)|+1).$$

This proves the Proposition 4.1. □

Remark 4.3. A careful inspectation of the proof of Proposition 4.1 shows that the constant $c', \delta, c_\delta, \epsilon(\delta)$, and $\bar{c}$ continuously depend on the 2-form $\sigma \in \Omega^2(M)$. In particular, Proposition 4.1 can be extended to families of symplectic forms.

Lemma 4.4 (Linear isoperimetric inequality)**.** Let $\sigma \in \Omega^2(N)$ be a weakly exact 2-form and $u_\sigma \sim 1$, then

$$\int_D \bar{q}^*\sigma \le C\left(\int_0^1 |\partial_t q|dt + 1\right)$$

where, $\bar{q}, D$ are the same as in Remark 3.4 and $C = C(N, g, \sigma, q_\nu)$.

Proof. The proof uses the same argument as in Lemma 2.4. Let $\widetilde{\bar{q}} : D \to \widetilde{N}$ be the lifting of $\bar{q}$ to the universal cover and $\theta \in \Omega^1(\widetilde{N})$ be a bounded primitive of $\widetilde{\sigma}$ as in Lemma 3.3. Then we get

$$\begin{aligned}
\int_D \bar{q}^*\sigma &= \int_D \widetilde{\bar{q}}^*\widetilde{\sigma} \\
&= \int_D \widetilde{\bar{q}}^* d\theta \\
&= \int_R \widetilde{q}^*\theta \\
&\le \left|\int_0^1 \widetilde{q}^*\theta\right| + \left|\int_0^1 \widetilde{q}_\nu^*\theta\right| + \left|\int_0^1 \underline{r}^*\theta\right| + \left|\int_0^1 \bar{r}^*\theta\right| \\
&\le \|\theta\|_\infty\left(\int_0^1 |\partial_t q|dt + \int_0^1 |\partial_t q_\nu|dt + \int_0^1 |\partial_t \underline{r}|dt + \int_0^1 |\partial_t \bar{r}|dt\right),
\end{aligned}$$

where R is a rectangle in $\widetilde{N}$ which consists of $\widetilde{q}$, $\widetilde{q}_\nu$, $\underline{r}$, and $\bar{r}$. Here, $\underline{r} : [0,1] \to \widetilde{N}$ is a path from $\widetilde{q}(0)$ to $\widetilde{q}_\nu(0)$ and $\bar{r} : [0,1] \to \widetilde{N}$ is a path from $\widetilde{q}(1)$ to $\widetilde{q}_\nu(1)$. Since $\int_D \bar{q}^*\sigma$ does not depend on the choice of D, we may assume that $\underline{r}$, $\bar{r}$ are length minimizing curves on $\widetilde{N}$. This implies that $\underline{r}$, $\bar{r}$ are geodesics contained in a fundamental domain in $\widetilde{N}$ or

$$\int_0^1 |\partial_t \underline{r}|dt \le \operatorname{diam}(N), \qquad \int_0^1 |\partial_t \bar{r}|dt \le \operatorname{diam}(N).$$

Set $C = \max\left\{\|\theta\|_\infty, 2\|\theta\|_\infty \int_0^1 |\partial_t q_\nu|dt, 4\|\theta\|_\infty \operatorname{diam}(N)\right\}$ then we get the conclusion. □

Remark 4.5. Note that C converges to 0 as $|\sigma|_g \to 0$.

Remark 4.6. If we consider the family of symplectic forms on T^*N

$$\omega_s = \omega_\mu + \beta(s)\tau^*\sigma \in \Omega^{\mathfrak{M}}(\Sigma_k), \qquad \forall s \in \mathbb{R},$$

where $\beta(s) \in C^\infty(\mathbb{R}, [0,1])$ is a cut-off function satisfying $\beta(s) = 1$ for $s \geq 1$, $\beta(s) = 0$ for $s \leq 0$, and $0 \leq \dot{\beta}(s) \leq 2$, then we obtain the estimate

$$\begin{aligned}\left|\int_D \overline{v}^* \dot{\omega}_s\right| &\leq \left|\int_D \overline{v}^* \dot{\beta}(s) \tau^* \sigma\right| \\ &= \dot{\beta}(s) \left|\int_D \overline{v}^* \tau^* \sigma\right| \\ &\leq C\dot{\beta}(s) \left(\int_{S^1} |\partial_t v(t)| dt + 1\right),\end{aligned}$$

for some $C = C(N, g, \sigma, q_\nu)$ given in Lemma 4.4.

Proposition 4.7. *Let $w = (v, \eta) \in C^\infty(\mathbb{R} \times S^1, T^*N) \times C^\infty(\mathbb{R}, \mathbb{R})$ be a gradient flow line of*

$$\mathcal{A}_{\omega(s)}(v, \eta) := \mathcal{A}_{H,\omega_s}(v, \eta) = \int_D \overline{v}^* \omega_s - \eta \int_0^1 H(x(t)) dt$$

i.e. a solution of

$$\left.\begin{array}{c}\partial_s v + J_{t,s}(v) \left(\partial_t v - \eta X_H^{\omega_s}(v)\right) = 0 \\ \partial_s \eta + \int_0^1 H(v(t)) dt = 0\end{array}\right\} \tag{4.5}$$

$$\lim_{s \to -\infty} w(s) = w_- \in \mathrm{Crit}\mathcal{A}_{\omega(0)}, \qquad \lim_{s \to \infty} w(s) = w_+ \in \mathrm{Crit}\mathcal{A}_{\omega(1)},$$

where ω_s is same as in Remark 4.6. If $|\sigma|_g$ is sufficiently small then the L^∞-norm of η is uniformly bounded in terms of a constant which only depends on w_-, w_+.

Proof. We prove the proposition in three steps.

Step1 : Let us first *bound the energy of w in terms of $\|\eta\|_\infty$.*

$$\begin{aligned}E(w) &= \int_{-\infty}^{\infty} \|\partial_s w\|_s^2 ds \\ &= \int_{-\infty}^{\infty} \langle \partial_s w, \nabla \mathcal{A}_{\omega(s)}(w) \rangle_s ds \\ &= \int_{-\infty}^{\infty} \frac{d}{ds} \mathcal{A}_{\omega(s)}(w) ds - \int_{-\infty}^{\infty} \dot{\mathcal{A}}_{\omega(s)}(w) ds \\ &= \mathcal{A}_{\omega(1)}(w_+) - \mathcal{A}_{\omega(0)}(w_-) - \int_{-\infty}^{\infty} \dot{\mathcal{A}}_{\omega(s)}(w) ds.\end{aligned} \tag{4.6}$$

We estimate the third term by

$$\begin{aligned}\left|\int_{-\infty}^{\infty} \dot{\mathcal{A}}_{\omega(s)}(w) ds\right| &\leq \int_{-\infty}^{\infty} \left|\dot{\mathcal{A}}_{\omega(s)}(w)\right| ds \\ &= \int_{-\infty}^{\infty} \dot{\beta}(s) \left|\int_D \overline{v}^* \tau^* \sigma\right| ds \\ &\leq \int_{-\infty}^{\infty} \dot{\beta}(s) C \left(\int_{S^1} |\partial_t v|_{t,s} dt + 1\right) ds,\end{aligned} \tag{4.7}$$

where C is the isoperimetric constant in Remark 4.6 and $|\cdot|_{t,s}$ is the norm on T^*N induced by the Riemannian metric $\omega_s(\cdot, J_{t,s} \cdot)$. From the gradient flow equation (4.5) we get

$$\partial_t v = J_{t,s}(v) \partial_s v + \eta X_H^{\omega_s}(v).$$

By putting this into (4.7), we then obtain

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\dot{\mathcal{A}}_{\omega(s)}(w)\right|ds &\leq \int_{-\infty}^{\infty}\dot{\beta}(s)C\left(\int_{S^1}|\partial_t v|_{t,s}dt+1\right)ds\\
&\leq \int_{-\infty}^{\infty}\dot{\beta}(s)C\left(\int_{S^1}|J_{t,s}(v)\partial_s v+\eta X_H^{\omega_s}(v)|_{t,s}dt+1\right)ds\\
&\leq \int_{-\infty}^{\infty}\underbrace{\dot{\beta}(s)}_{\leq 2}C\left(\int_{S^1}\left(|\partial_s v|_{t,s}+|\eta|\;|X_H^{\omega_s}(v)|_{t,s}\right)dt+1\right)ds\\
&\leq 2C\int_0^1\left(\int_{S^1}\left(|\partial_s v|_{t,s}^2+1+|\eta|\;|X_H^{\omega_s}(v)|_{t,s}\right)dt+1\right)ds\\
&\leq 2CE(v)+4C+2C\|\eta\|_\infty c''\\
&\leq 2CE(w)+4C+2C\|\eta\|_\infty c'',\end{aligned}$$

where $c''=\max_{\substack{s\in[0,1]\\ v\in T^*N}}|X_H^{\omega_s}(v)|_{t,s}$. Note that the maximun is attained, since by the assumption dH has compact support. Now by substituting the above equation into (4.6), we get

$$\begin{aligned}E(w)&=\mathcal{A}_{\omega(1)}(w_+)-\mathcal{A}_{\omega(0)}(w_-)-\int_{-\infty}^{\infty}\dot{\mathcal{A}}_{\omega(s)}(w)ds\\
&\leq \mathcal{A}_{\omega(1)}(w_+)-\mathcal{A}_{\omega(0)}(w_-)+2CE(w)+4C+2C\|\eta\|_\infty c''\end{aligned}\tag{4.8}$$

By choosing $\sigma\in\Omega^2(M)$ with sufficiently small norm, we may assume that the isoperimetric constant C is less than $\frac{1}{4}$. For simplicity, set $\Delta=\mathcal{A}_{\omega(1)}(w_+)-\mathcal{A}_{\omega(0)}(w_-)$, then we get

$$\begin{aligned}E(w)&\leq 2\mathcal{A}_{\omega(1)}(w_+)-2\mathcal{A}_{\omega(0)}(w_-)+8C+4C\|\eta\|_\infty c''\\
&=2\Delta+8C+4C\|\eta\|_\infty c''.\end{aligned}\tag{4.9}$$

This finishes Step1.

Step2 : Let ϵ be as in Proposition 4.1 and Remark 4.3. For $l\in\mathbb{R}$ let $\tau(l)\geq 0$ be defined by

$$\tau(l):=\inf\{\tau\geq 0:\|\nabla\mathcal{A}_{\omega(s)}((v,\eta)(l+\tau))\|_s<\epsilon\}.$$

In this step we *bound* $\tau(l)$ *in terms of* $\|\eta\|_\infty$ *for all* $l\in\mathbb{R}$. Namely

$$\begin{aligned}E(w)&=\int_{-\infty}^{\infty}\|\partial_s w\|_s^2ds\\
&=\int_{-\infty}^{\infty}\|\nabla\mathcal{A}_{\omega(s)}\|_s^2ds\\
&\geq\int_l^{l+\tau(l)}\underbrace{\|\nabla\mathcal{A}_{\omega(s)}\|_s^2}_{\geq\epsilon^2}ds\\
&\geq\epsilon^2\tau(l)\end{aligned}$$

Step1 and the above estimate finish Step2.

Step3 : *We prove the proposition.*
First set

$$\|H\|_\infty=\max_{x\in T^*N}|H(x)|,\quad K=\max\{-\mathcal{A}_{\omega(0)}(w_-),\mathcal{A}_{\omega(1)}(w_+)\}.$$

We estimate using Proposition 4.1, Step1 and Step2.

$$
\begin{aligned}
|\eta(l)| &\leq |\eta(l+\tau(l))| + \left| \int_l^{l+\tau(l)} \dot{\eta}(s) ds \right| \\
&\leq |\eta(l+\tau(l))| + \left| \int_l^{l+\tau(l)} \int_0^1 H(v(t)) dt \ ds \right| \\
&\leq \overline{c}(|\mathcal{A}_{\omega(s)}(w(l+\tau(l)))| + 1) + \|H\|_\infty \tau(l) \\
&\leq \overline{c}\left(K + \int_{-\infty}^{\infty} \left| \dot{\mathcal{A}}_{\omega(s)} \right| ds + 1 \right) + \|H\|_\infty \tau(l) \\
&\leq \overline{c}\left(K + 2CE(w) + 4C + 2C\|\eta\|_\infty c'' + 1 \right) + \|H\|_\infty \frac{E(w)}{\epsilon^2} \\
&\leq \overline{c}\left(K + 4C\Delta + 16C^2 + 8C^2\|\eta\|_\infty c'' + 4C + 2C\|\eta\|_\infty c'' + 1 \right) \\
&\quad + \frac{\|H\|_\infty}{\epsilon^2}(2\Delta + 8C + 4C\|\eta\|_\infty c'') \\
&= \underbrace{\left(8\overline{c}c''C + 2\overline{c}c'' + \frac{4c''\|H\|_\infty}{\epsilon^2} \right)}_{=:K'} C\|\eta\|_\infty \\
&\quad + \underbrace{\overline{c}K + 4\overline{c}C\Delta + 16\overline{c}C^2 + 4\overline{c}C + \overline{c} + \frac{2\|H\|_\infty\Delta}{\epsilon^2} + \frac{8C\|H\|_\infty}{\epsilon^2}}_{=:K''}.
\end{aligned}
$$

Since the above estimate holds for all $l \in \mathbb{R}$

$$
\|\eta\|_\infty \leq K'C\|\eta\|_\infty + K''.
$$

We can achieve that the *isoperimetric constant* C satisfies

$$
C \leq \min\{1/4, 1/2K'\} \tag{4.10}
$$

by choosing $\sigma \in \Omega^2(M)$ with small norm. This proves the proposition. $\square$

Lemma 4.8. Assume that the isoperimetric constant C is sufficiently small, then the following holds true. Suppose that $w = (v, \eta) \in C^\infty(\mathbb{R} \times S^1, T^*N) \times C^\infty(\mathbb{R}, \mathbb{R})$ is a gradient flow line of the time dependent gradient $\nabla\mathcal{A}_{\omega(s)}$ which converges asymptotically $\lim_{s\to\pm} w(s) = w_\pm$ to critical points of $\mathcal{A}_{\omega(1)}, \mathcal{A}_{\omega(0)}$ respectively such that $a = \mathcal{A}_{\omega(0)}(w_-)$ and $b = \mathcal{A}_{\omega(1)}(w_+)$. Then the following assertions meet

(1) If $a \geq \frac{1}{9}$, then $b \geq \frac{a}{2}$;

(2) If $b \leq -\frac{1}{9}$, then $a \leq \frac{b}{2}$.

Proof. By the previous proposition, we obtained the following uniform bound of η

$$
\begin{aligned}
\|\eta\|_\infty &\leq 2K'' \\
&= 2\overline{c}K + 8\overline{c}C\Delta + 32\overline{c}C^2 + 8\overline{c}C + 2\overline{c} + \frac{4\|H\|_\infty\Delta}{\epsilon^2} + \frac{16C\|H\|_\infty}{\epsilon^2}.
\end{aligned}
$$

Moreover, since $E(w) \geq 0$ we obtain from (4.9) the inequality

$$
b \geq a - 4C - 2C\|\eta\|_\infty c''.
$$

By taking a small isoperimetric constant C satisfying

$$\begin{aligned} C\bar{c}c'' &\leq \frac{1}{32}; \\ C\left(2\bar{c}C + \frac{\|H\|_\infty}{\epsilon^2}\right)c'' &\leq \frac{1}{128}; \\ C\left(1 + 16\bar{c}c''C^2 + 4\bar{c}c''C + \bar{c}c'' + \frac{8c''C\|H\|_\infty}{\epsilon^2}\right) &\leq \frac{1}{144}; \end{aligned} \tag{4.11}$$

we now get

$$b \geq a - \frac{1}{8}K - \frac{1}{16}(b-a) - \frac{1}{36}, \tag{4.12}$$

where $K = \max\{-a, b\}$. To prove the assertion (1), we first consider the case

$$|b| \leq a, \qquad a \geq \frac{1}{9}.$$

In this case, we estimate

$$b \geq a - \frac{1}{8}a - \frac{1}{8}a - \frac{1}{36} = \frac{3}{4}a - \frac{1}{36} \geq \frac{a}{2}.$$

Hence to prove the assertion (1), it suffices to exclude the case

$$-b \geq a \geq \frac{1}{9}.$$

But in this case, (4.12) leads to a contradiction in the following way

$$b \geq \frac{1}{9} + \frac{1}{72} - \frac{1}{16}(b-a) - \frac{1}{36} \geq -\frac{1}{16}(b-a) > 0.$$

This proves the first assertion. To prove the assertion (2), we set

$$b' = -a, \qquad a' = -b.$$

We note that if (4.12) holds for a and b, it also holds for b' and a'. Hence we get from the assertion (1) the implication

$$-b \geq \frac{1}{9} \Longrightarrow -a \geq -\frac{b}{2}$$

which is equivalent to the assertion (2). This finishes the proof of the Lemma. □

Proof of Theorem 1.4. We now construct the continuation homomorphism

$$\Psi_{\omega_0}^{\omega_\sigma} : \mathbf{RFH}(\Sigma_k, \omega_0 = dp \wedge dq) \to \mathbf{RFH}(\Sigma_k, \omega_\sigma = dp \wedge dq + \tau^*\sigma)$$

for $\omega_0, \omega_\sigma \in \Omega^{\mathfrak{M}}_{\mathrm{reg}}(\Sigma_k)$. Similar as in Theorem 2.7, we first subdivide

$$\omega_s = \omega_0 + s(\omega_\sigma - \omega_0)$$

into small pieces. Let $\{\omega^i\}_{i=0}^N$ be a subdivision of ω_s satisfying

- $\omega^i = \omega_0 + d(i)\tau^*\sigma$, where $0 = d(0) < d(1) < \cdots < d(N) = 1$;
- $\mathcal{A}_{H,\omega^i} : \mathscr{L} \times \mathbb{R} \to \mathbb{R}$ is Morse-Bott, $\forall i = 0, 1, \ldots, N$;
- $C(M, g, (d(i+1) - d(i))\sigma, v_\nu)$ satisfies (4.10), (4.11), $\forall i = 0, 1, \ldots, N-1$.

Let $\omega_s^i = \omega^i + \beta(s)(\omega^{i+1} - \omega^i)$ be a homotopy between ω^i and ω^{i+1}. First we construct the following continuation map

$$\widetilde{\Psi}_{\omega^i}^{\omega^{i+1}} : \mathbf{RFH}(\Sigma_k, \omega^i) \to \mathbf{RFH}(\Sigma_k, \omega^{i+1}).$$

Since the action functional $\mathcal{A}_{H,\omega^i}$ is Morse-Bott, the construction is given by counting gradient flow lines with cascades as in the Morse-Bott homology. Let us choose Morse functions h^i on $\mathrm{Crit}(\mathcal{A}_{H,\omega^i})$. We then define a map

$$\Psi_{\omega^i\ *}^{\omega^{i+1}} : \mathbf{RFC}_*(\Sigma_k, \omega^i) \to \mathbf{RFC}_*(\Sigma_k, \omega^{i+1})$$

given by

$$\Psi_{\omega^i}^{\omega^{i+1}}(w_-) = \sum_{\mu(w_+)=\mu(w_-)} \#_2 \mathcal{M}_{\omega^i}^{\omega^{i+1}}(w_-, w_+) w_+,$$

where $w_- \in \mathrm{Crit}(h^i)$, $w_+ \in \mathrm{Crit}(h^{i+1})$ and $\#_2$ denotes the $\mathbb{Z}_2$-counting. Here,

$$\begin{aligned}
\widehat{\mathcal{M}}_{\omega^i,m}^{\omega^{i+1}}(w_-, w_+) &= \{w \mid w \text{ is a flow line with } m\text{-cascades from } w_- \text{ to } w_+\};\\
\mathcal{M}_{\omega^i,m}^{\omega^{i+1}}(w_-, w_+) &= \widehat{\mathcal{M}}_{\omega^i,m}^{\omega^{i+1}}(w_-, w_+)/\mathbb{R}^m;\\
\mathcal{M}_{\omega^i}^{\omega^{i+1}}(w_-, w_+) &= \bigcup_{m\in\mathbb{N}_0} \mathcal{M}_{\omega^i,m}^{\omega^{i+1}}(w_-, w_+).
\end{aligned}$$

The main issue of this construction is also the uniform bound of $E(w)$. As in the Morse-Bott homology situation, it suffices to check that each gradient flow line has a uniform energy bound. For this reason, we now only consider the following uniform energy bound. Let

$$w' = (v', \eta') \in C^\infty(\mathbb{R} \times S^1, T^*N) \times C^\infty(\mathbb{R}, \mathbb{R})$$

be a gradient flow line of

$$\mathcal{A}_{\omega_s^i}(v, \eta) = \int_D \overline{v}^* \omega_s^i - \eta \int_0^1 H(x(t))dt$$

i.e. a solution of

$$\left.\begin{aligned}
\partial_s v + J_{t,s}(v)\left(\partial_t v - \eta X_H^{\omega_s^i}(v)\right) = 0\\
\partial_s \eta + \textstyle\int_0^1 H(x(t))dt = 0
\end{aligned}\right\}$$

$$\lim_{s\to-\infty} w'(s) = w'_- \in \mathrm{Crit}\mathcal{A}_{\omega^i}, \qquad \lim_{s\to\infty} w'(s) = w'_+ \in \mathrm{Crit}\mathcal{A}_{\omega^{i+1}}.$$

To achieve a uniform energy bound of w', let us recall the equation (4.8) in Proposition 4.7

$$E(w') \le \mathcal{A}_{\omega^{i+1}}(w'_+) - \mathcal{A}_{\omega^i}(w'_-) + 2CE(w') + 4C + 2C\|\eta'\|_\infty c''.$$

Since the isoperimetric constant C is less than $\min\{1/4, 1/2K'\}$, we get the following uniform bound of the Lagrangian multiplier η'

$$\|\eta'\|_\infty \le 2K''$$

and

$$\begin{aligned}
E(w') &\le 2\Delta + 8C + 4C\|\eta'\|_\infty c''\\
&\le 2\Delta + 8C + 8Cc''K'',
\end{aligned}$$

where the coefficients are the same as in Proposition 4.7. Hence we conclude $E(w')$ is uniformly bounded.

Now, by virtue of Lemma 4.8, we obtain for $a \leq -\frac{1}{9}$ and $b \geq \frac{1}{9}$ maps

$$\Psi_{\omega_i}^{\omega_{i+1}(a,b)} : \mathbf{RFC}^{(\frac{a}{2},b)}(\Sigma_k, \omega_i) \to \mathbf{RFC}^{(a,\frac{b}{2})}(\Sigma_k, \omega_{i+1})$$

defined by counting gradient flow lines of the time dependent Rabinowitz action functional. Since the continuation map $\Psi_{\omega_i}^{\omega_{i+1}(a,b)}$ commutes with the boundary operators, this induces the following homomorphism on homology level.

$$\widetilde{\Psi}_{\omega_i}^{\omega_{i+1}(a,b)} : \mathbf{RFH}^{(\frac{a}{2},b)}(\Sigma_k, \omega_i) \to \mathbf{RFH}^{(a,\frac{b}{2})}(\Sigma_k, \omega_{i+1})$$

By taking the inverse and direct limit as follows

$$\mathbf{RFH}_*(\Sigma_k, \omega_i) = \lim_{b\to\infty} \lim_{a\to-\infty} \mathbf{RFH}_*^{(a,b)}(\Sigma_k, \omega_i),$$

we obtain

$$\widetilde{\Psi}_{\omega_i}^{\omega_{i+1}} : \mathbf{RFH}(\Sigma_k, \omega_i) \to \mathbf{RFH}(\Sigma_k, \omega_{i+1}).$$

Similar in usual Floer homology, we can define the continuation homomorphism by juxtaposition

$$\widetilde{\Psi}_{\omega_0}^{\omega_\sigma} : \mathbf{RFH}(\Sigma_k, \omega_0) \to \mathbf{RFH}(\Sigma_k, \omega_\sigma)$$

$$\widetilde{\Psi}_{\omega_0}^{\omega_\sigma} = \widetilde{\Psi}_{\omega^{N-1}}^{\omega^N} \circ \cdots \circ \widetilde{\Psi}_{\omega^1}^{\omega^2} \circ \widetilde{\Psi}_{\omega^0}^{\omega^1}.$$

In a similar way, we can construct

$$\widetilde{\Psi}_{\omega_\sigma}^{\omega_0} : \mathbf{RFH}(\Sigma_k, \omega_\sigma) \to \mathbf{RFH}(\Sigma_k, \omega_0),$$

by following the homotopy backwards. By a homotopy-of-homotopies argument, we conclude $\widetilde{\Psi}_{\omega_\sigma}^{\omega_0} \circ \widetilde{\Psi}_{\omega_0}^{\omega_\sigma} = \mathrm{id}_{\mathbf{RFH}(\Sigma_k,\omega_0)}$ and $\widetilde{\Psi}_{\omega_0}^{\omega_\sigma} \circ \widetilde{\Psi}_{\omega_\sigma}^{\omega_0} = \mathrm{id}_{\mathbf{RFH}(\Sigma_k,\omega_\sigma)}$. Therefore $\widetilde{\Psi}_{\omega_0}^{\omega_1}$ is an isomorphism with inverse $\widetilde{\Psi}_{\omega_1}^{\omega_0}$. □

Remark 4.9. In the above proof the *linear isoperimetric inequality* is crucial. One can easily check that the Proof of Theorem 1.4 does not work for $u_\sigma(t) \gtrsim 1$.

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